THE STRUCTURE OF DRINFELD MODULAR FORMS OF LEVEL $\Gamma_0(T)$ AND APPLICATIONS

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ABSTRACT. In this article, we describe the structure of the R-algebra of Drinfeld modular forms $M(\Gamma_0(T))_R$ (resp., $M^0(\Gamma_0(T))_R$) of level $\Gamma_0(T)$ and the structure of mod-$\mathfrak{p}$ reduction of $M_{\mathfrak{p}}^0(\Gamma_0(T))$ for $\mathfrak{p} \neq (T)$. As a result, we are able to study the properties of the weight filtration for $M_{k,l}(\Gamma_0(T))$. Finally, we prove a result on mod-$\mathfrak{p}$ congruences for Drinfeld modular forms of level $\Gamma_0(\mathfrak{p}T)$ for $\mathfrak{p} \neq (T)$.

1. INTRODUCTION

It is well-known that the $\mathbb{C}$-algebra of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ is generated by E_4 and E_6, where E_4, E_6 denote the Eisenstein series of weight 4, 6, respectively. As a consequence, Serre, Swinnerton-Dyer studied the properties of the weight filtration for $\mathrm{SL}_2(\mathbb{Z})$ (cf. [Ser71], [Swi73]). Similarly, the $\mathbb{C}$-algebra of modular forms for $\Gamma_0(2)$ is generated by $E_2^*(q) := E_2(q) - 2E_2(q^2)$, E_4. There are some important works, in the literature, about the structure of the $\mathbb{C}$-algebra of classical modular forms for higher levels (cf. [Rus14], [VZ] for more details).

It is of natural interest to understand the structure of the C-algebra of Drinfeld modular forms of level $\Gamma_0(\mathfrak{n})$. Such a study has been initiated by Gekeler in [Gek88], where he proved that the C-algebra $M(\mathrm{GL}_2(A))$, Drinfeld modular forms of level $\mathrm{GL}_2(A)$, is isomorphic to the polynomial ring in two variables $C[X,Y]$. Hence, to each $f \in M(\mathrm{GL}_2(A))$, one can attach a unique polynomial in $C[X,Y]$. Using these polynomials and the mod-$\mathfrak{p}$ reduction of the ring $M_{\mathfrak{p}}(\mathrm{GL}_2(A))$, Drinfeld modular forms of level $\mathrm{GL}_2(A)$ with $\mathfrak{p}$-integral u-expansion at ∞, Vincent ([Vin10]) studied the properties of the weight filtration for $M_{k,l}(\mathrm{GL}_2(A))$. To the best of authors knowledge, the analogues of these results for higher levels is unknown.

In this article, we show that R-algebra $M(\Gamma_0(T))_R$, Drinfeld modular forms of level $\Gamma_0(T)$ with coefficients in R, is generated by Δ_W, Δ_T, E_T and it is the quotient of a polynomial ring in 3-variables $R[U,V,Z]$. Hence, every $f \in M(\Gamma_0(T))_R$ can be expressed as a polynomial in $R[U,V,Z]$, but it is not unique.

The novelty in our work is to show that the properties of the weight filtration for $M_{k,l}(\Gamma_0(T))$ depends only on the structure of $M^0(\Gamma_0(T))_R$, Drinfeld modular forms of type 0, level $\Gamma_0(T)$ with coefficients in R. So, we describe the structure of $M^0(\Gamma_0(T))_R$ and show that it is a polynomial ring in two variables $R[U,V]$. Hence, for $f \in M_{k,0}(\Gamma_0(T))_R$, one can attach a unique polynomial $\varphi_f(U,V) \in$

$R[U, V]$ such that $f = \varphi_f(\Delta_W, \Delta_T)$. Using these polynomials and the mod-$\mathfrak{p}$ reduction of the ring $M_{\mathfrak{p}}^0(\Gamma_0(T))$, Drinfeld modular forms of type 0, level $\Gamma_0(T)$ with $\mathfrak{p}$-integral u-expansion at ∞, we study the properties of the weight filtration for $M_{k,l}(\Gamma_0(T))$.

Finally, as an application, we shall prove a result on mod-$\mathfrak{p}$ congruences for Drinfeld modular forms of level $\Gamma_0(\mathfrak{p}T)$ for $\mathfrak{p} \neq (T)$.

1.1. **Notations:** Throughout the article, we fix to use the following notations.

Let p be an odd prime number and $q = p^r$ for some $r \in \mathbb{N}$. Let $\mathbb{F}_q$ denote the finite field of order q. We set $A := \mathbb{F}_q[T]$ and $K := \mathbb{F}_q(T)$. Let $K_\infty = \mathbb{F}_q((\frac{1}{T}))$ be the completion of K with respect to the infinite place ∞ (corresponding to $\frac{1}{T}$-adic valuation) and the completion of an algebraic closure of K_∞ is denoted by C.

- Let $k \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{Z}/(q-1)\mathbb{Z}$ such that $k \equiv 2l \pmod{q-1}$. Let $0 \leq l \leq q - 2$ be a lift of $l \in \mathbb{Z}/(q-1)\mathbb{Z}$. By abuse of notation, we continue to write l for the integer as well as its class. Then, we define $r_{k,l} := \frac{k-2l}{q-1}$.
- Let R be a ring such that $A \subseteq R \subseteq C$.
- $\mathfrak{p}$ denotes a prime ideal of A generated by a monic irreducible polynomial π of degree d such that $\mathfrak{p} \neq (T)$.

2. Preliminaries

In this section, we recall some basic theory of Drinfeld modular forms (cf. [Gos80], [Gos80a], [Gek88], [GR96] for more details).

Let $L = \tilde{\pi}A \subseteq C$ be the A-lattice of rank 1, corresponding to the rank 1 Drinfeld module given by $\rho_T = TX + X^q$, which is also known as the Carlitz module, where $\tilde{\pi} \in K_\infty(\sqrt[q-1]{-T})$ is defined up to a $(q-1)$-th root of unity. Any $x \in K_\infty^\times$ has the unique expression $x = \zeta_x\left(\frac{1}{T}\right)^{v_\infty(x)} u_x$, where $\zeta_x \in \mathbb{F}_q^\times$, and $v_\infty(u_x - 1) \geq 0$ (v_∞ is the valuation at ∞). The Drinfeld upper half-plane $\Omega = C - K_\infty$ has a rigid analytic structure. The group $\mathrm{GL}_2(K_\infty)$ acts on Ω via fractional linear transformations. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\infty)$ and $f : \Omega \to C$, we define $f|_{k,l}\gamma := \zeta_{\det \gamma}^l \left(\frac{\det \gamma}{\zeta_{\det(\gamma)}}\right)^{k/2}(cz+d)^{-k}f(\gamma z)$. For an ideal $\mathfrak{n} \subseteq A$, we define the congruence subgroup $\Gamma_0(\mathfrak{n})$ of $\mathrm{GL}_2(A)$ by $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A) : c \in \mathfrak{n}\}$.

Definition 2.1. Let $k \in \mathbb{N} \cup \{0\}, l \in \mathbb{Z}/(q-1)\mathbb{Z}$. A rigid holomorphic function $f : \Omega \to C$ is said to be a Drinfeld modular form of weight k, type l for $\Gamma_0(\mathfrak{n})$ if

(1) $f|_{k,l}\gamma = f$, $\forall \gamma \in \Gamma_0(\mathfrak{n})$,
(2) f is holomorphic at the cusps of $\Gamma_0(\mathfrak{n})$.

The space of Drinfeld modular forms of weight k, type l for $\Gamma_0(\mathfrak{n})$ is denoted by $M_{k,l}(\Gamma_0(\mathfrak{n}))$. Furthermore, if f vanishes (resp., vanishes at least twice) at the cusps of $\Gamma_0(\mathfrak{n})$, then we say f is a Drinfeld cusp (resp., doubly cuspidal) form of weight k, type l for $\Gamma_0(\mathfrak{n})$. These spaces are denoted by $M_{k,l}^1(\Gamma_0(\mathfrak{n}))$ (resp., $M_{k,l}^2(\Gamma_0(\mathfrak{n}))$).

Note that, if $k \not\equiv 2l \pmod{q-1}$ then $M_{k,l}(\Gamma_0(\mathfrak{n})) = \{0\}$. So, without loss of generality, we assume that $k \equiv 2l \pmod{q-1}$.

Let $u(z) := \frac{1}{e_L(\tilde{\pi}z)}$, where $e_L(z) := z\prod_{0 \neq \lambda \in L}(1 - \frac{z}{\lambda})$ is the exponential function attached to the lattice L, which is associated with the Carlitz module. Then, each Drinfeld modular form $f \in M_{k,l}(\Gamma_0(\mathfrak{n}))$ has a unique u-expansion at ∞ given by $f = \sum_{i=0}^{\infty} a_f(i)u^i$. Since $\left(\begin{smallmatrix} \zeta & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{n})$ for $\zeta \in \mathbb{F}_q^{\times}$, condition (1) of Definition 2.1 implies $a_f(i) = 0$ if $i \not\equiv l \pmod{q-1}$. Hence, the u-expansion of f at ∞ can be written as $\sum_{0 \leq i \equiv l \mod (q-1)} a_f(i)u^i$. Note that any Drinfeld modular form of type > 0 is automatically a cusp form.

We define $M_{k,l}(\Gamma_0(\mathfrak{n}))_R$ to be the space of Drinfeld modular forms f of weight k, type l for $\Gamma_0(\mathfrak{n})$ such that the u-expansion of f at ∞ belongs to $R[[u]]$ (or we simply say "with coefficients in R"). For any $0 \neq f \in M_{k,l}(\Gamma_0(\mathfrak{n}))_R$, we define $f^0 := 1$.

We end this discussion by introducing the notion of $\mathfrak{p}$-adic valuation of f. This makes it easier to define a congruence between two Drinfeld modular forms.

Definition 2.2. Let $f = \sum_{n \geq 0} a_f(n)u^n$ be a formal u-series in $K[[u]]$. We define $v_{\mathfrak{p}}(f) := \inf_n v_{\mathfrak{p}}(a_f(n))$, where $v_{\mathfrak{p}}(a_f(n))$ is the $\mathfrak{p}$-adic valuation of $a_f(n)$. We say f has a $\mathfrak{p}$-integral u-expansion if $v_{\mathfrak{p}}(f) \geq 0$.

Definition 2.3 (Congruence). Let $f = \sum_{n \geq 0} a_f(n)u^n$ and $g = \sum_{n \geq 0} a_g(n)u^n$ be two u-formal series in $K[[u]]$. We say that $f \equiv g \pmod{\mathfrak{p}}$ if $v_{\mathfrak{p}}(f - g) \geq 1$.

2.1. **Examples.** We now give some examples of Drinfeld modular forms.

Example 2.4 ([Gos80], [Gek88]). Let $d \in \mathbb{N}$. For $z \in \Omega$, the function

$$g_d(z) := (-1)^{d+1}\tilde{\pi}^{1-q^d}L_d \sum_{\substack{a,b \in \mathbb{F}_q[T] \\ (a,b) \neq (0,0)}} \frac{1}{(az+b)^{q^d-1}}$$

is a Drinfeld modular form of weight $q^d - 1$, type 0 for $\mathrm{GL}_2(A)$, where $\tilde{\pi}$ is the Carlitz period and $L_d := (T^q - T) \cdots (T^{q^d} - T)$ is the least common multiple of all monic polynomials of degree d. We refer to g_d as an Eisenstein series.

Example 2.5 ([Gos80a], [Gek88]). For $z \in \Omega$, the function

$$\Delta(z) := (T - T^{q^2})\tilde{\pi}^{1-q^2}E_{q^2-1} + (T^q - T)^q\tilde{\pi}^{1-q^2}(E_{q-1})^{q+1},$$

is a Drinfeld cusp form of weight $q^2 - 1$, type 0 for $\mathrm{GL}_2(A)$, where $E_k(z) = \sum_{(0,0) \neq (a,b) \in A^2} \frac{1}{(az+b)^k}$. The u-expansion of Δ at ∞ is given by $-u^{q-1} + \cdots$.

Example 2.6 ([Gek88]). For $z \in \Omega$, the function

$$h(z) := \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H \backslash \mathrm{GL}_2(A)} \frac{\det \gamma . u(\gamma z)}{(cz+d)^{q+1}},$$

is a Drinfeld cusp form of weight $q + 1$, type 1 for $\mathrm{GL}_2(A)$, where $H = \left\{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(A)\right\}$. The u-expansion of h at ∞ is given by $-u - u^{(q-1)^2+1} + \cdots$.

We end this section by introducing an important function E, which is not modular. In [Gek88], Gekeler defined the function $E(z) := \frac{1}{\tilde{\pi}}\sum_{\substack{a \in \mathbb{F}_q[T] \\ a \text{ monic}}}\left(\sum_{b \in \mathbb{F}_q[T]} \frac{a}{az+b}\right)$ which is analogous to the Eisenstein series of weight 2 over $\mathbb{Q}$. Using E, one can construct the Drinfeld modular form

$$E_T(z) := E(z) - TE(Tz) \in M_{2,1}(\Gamma_0(T)). \tag{2.1}$$

The u-expansion of E_T at ∞ is given by $u - Tu^q + \cdots \in A[[u]]$ (cf. [DK1, Proposition 3.3]). Since $\dim_C M_{2,1}(\Gamma_0(T)) = 1$, we have $M_{2,1}(\Gamma_0(T)) = \langle E_T \rangle$.

3. Statements of the main results

In this section, we state the main results of this article (cf. Theorem 4.4, Theorem 4.7, Theorem 4.13).

3.1. **On the structure of** $M(\Gamma_0(T))_R$**:** Let $M(\Gamma_0(\mathfrak{n}))_R := \oplus_{k,l} M_{k,l}(\Gamma_0(\mathfrak{n}))_R$ be the graded R-algebra of Drinfeld modular forms of level $\Gamma_0(\mathfrak{n})$ with coefficients in R. In [Gek88, Theorem 5.13], Gekeler described the C-algebra structure of $M(\mathrm{GL}_2(A))$. In fact, he proved $M(\mathrm{GL}_2(A)) = C[g_1, h]$, where $g_1 \in M_{q-1,0}(\mathrm{GL}_2(A))$, $h \in M_{q+1,1}(\mathrm{GL}_2(A))$ are as in Example 2.4, Example 2.6, respectively. In particular, any $f \in M_{k,l}(\mathrm{GL}_2(A))$ can be expressed as a unique isobaric polynomial in g_1 and h. To the best of authors knowledge, the structure of the algebra of Drinfeld modular forms for general $\Gamma_0(\mathfrak{n})$ is unknown. In § 4.1, we describe the structure of $M(\Gamma_0(T))_R$. More precisely, we prove

Theorem 3.1 (Theorem 4.4 in the text)**.** *The R-algebra $M(\Gamma_0(T))_R$ is generated by Δ_W, Δ_T and E_T, where Δ_W, Δ_T and E_T are as in (4.1) and (2.1), respectively. Moreover, the surjective map $\vartheta(R) : R[U, V, Z] \twoheadrightarrow M(\Gamma_0(T))_R$ defined by $U \to \Delta_W, V \to \Delta_T, Z \to E_T$ induces an isomorphism*

$$R[U, V, Z]/(UV - Z^{q-1}) \cong M(\Gamma_0(T))_R.$$

An immediate consequence of Theorem 3.1 and the well-known results from commutative algebra (cf. [Mat80]) would imply the following corollary.

Corollary 3.2. *For any ring R with $A \subseteq R \subseteq C$ we have $M(\Gamma_0(T))_R \cong R \otimes_A M(\Gamma_0(T))_A$. In fact, the Krull dimension of $M(\Gamma_0(T))_C$ is 2.*

3.2. **On the structure of** $M^0(\Gamma_0(T))_R$**:** Let $M^0(\Gamma_0(T))_R := \oplus_{k=0}^{\infty} M_{k,0}(\Gamma_0(T))_R$ denote the graded R-algebra of Drinfeld modular forms f of type 0, level $\Gamma_0(T)$ with coefficients in R. In [Cor97, Proposition 4.6.2], Cornelissen described the graded C-algebra structure of $M^0(\Gamma_0(T))$ and showed that $M^0(\Gamma_0(T)) = C[g_1(z), g_1(Tz)]$. However, these generators do not describe the A-structure, i.e., $M^0(\Gamma_0(T))_A \supsetneq A[g_1(z), g_1(Tz)]$. In §4.2, we specify a set of generators for the graded A-algebra $M^0(\Gamma_0(T))_A$. More precisely, we prove:

Theorem 3.3 (Theorem 4.7 in the text)**.** *The map*

$$\vartheta_0(R) : R[U, V] \longrightarrow M^0(\Gamma_0(T))_R$$

defined by $(U, V) \to (\Delta_W, \Delta_T)$ is an isomorphism. In particular, the R-algebra $M^0(\Gamma_0(T))_R$ is generated by Δ_W, Δ_T, i. e., $M^0(\Gamma_0(T))_R = R[\Delta_W, \Delta_T]$.

An immediate consequence of Theorem 3.3 and the well-known results from commutative algebra (cf. [Mat80]) would imply that

Corollary 3.4. *For any ring R with $A \subseteq R \subseteq C$, we have $M^0(\Gamma_0(T))_R \cong R \otimes_A M^0(\Gamma_0(T))_A$. Moreover, if R is a Noetherian local ring, then the Krull dimension of $M^0(\Gamma_0(T))_R$ is $\dim(R) + 2$.*

3.3. **On the mod-$\mathfrak{p}$ reduction:** Let $M_{\mathfrak{p}}(\mathrm{GL}_2(A))$ denote the space of Drinfeld modular forms for $\mathrm{GL}_2(A)$ with $\mathfrak{p}$-integral u-expansion at ∞. Define

$$\overline{M_{\mathfrak{p}}}(\mathrm{GL}_2(A)) := \{\overline{f} \in \mathbb{F}_{\mathfrak{p}}[[u]] \mid \exists f \in M_{\mathfrak{p}}(\mathrm{GL}_2(A)) \text{ such that } f \equiv \overline{f} \pmod{\mathfrak{p}}\},$$

where $\mathbb{F}_{\mathfrak{p}} := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ is the localization of A at $\mathfrak{p}$. The map $\epsilon : \mathbb{F}_{\mathfrak{p}}[X, Y] \to \overline{M_{\mathfrak{p}}}(\mathrm{GL}_2(A))$ defined by $X \to \overline{g}_1$, $Y \to \overline{h}$ is surjective and $\ker \epsilon = (\overline{A}_d(X, Y) - 1)$, where $A_d(X, Y)$ is the unique isobaric polynomial attached to g_d and "$-$" denotes the reduction mod $\mathfrak{p}$ (cf. [Gek88, Corollary 12.4]). In §4, we extend this result to Drinfeld modular forms of type 0 for $\Gamma_0(T)$.

Let $M_{\mathfrak{p}}^0(\Gamma_0(T))$ denote the space of Drinfeld modular forms of type 0 for $\Gamma_0(T)$ with $\mathfrak{p}$-integral u-expansion at ∞. We define

$$\overline{M_{\mathfrak{p}}^0}(\Gamma_0(T)) := \{\overline{f} \in \mathbb{F}_{\mathfrak{p}}[[u]] \mid \exists f \in M_{\mathfrak{p}}^0(\Gamma_0(T)) \text{ such that } f \equiv \overline{f} \pmod{\mathfrak{p}}\}.$$

Theorem 3.5 (Theorem 4.13 in text). *The surjective map* $\iota : \mathbb{F}_{\mathfrak{p}}[U, V] \twoheadrightarrow \overline{M_{\mathfrak{p}}^0}(\Gamma_0(T))$ *defined by* $U \to \overline{\Delta}_W$ *and* $V \to \overline{\Delta}_T$ *induces an isomorphism*

$$\mathbb{F}_{\mathfrak{p}}[U, V]/(\overline{\phi}_d(U, V) - 1) \cong \overline{M_{\mathfrak{p}}^0}(\Gamma_0(T)),$$

where $\phi_d(U, V)$ *is the unique isobaric polynomial attached to* g_d, *i.e.,* $g_d = \phi_d(\Delta_W, \Delta_T)$.

3.4. **Weight filtration and its properties:** By Theorem 3.1, for $f \in M_{k,l}(\Gamma_0(T))_R$, there exists a polynomial $\eta(U, V, Z) \in R[U, V, Z]$ such that $f = \eta(\Delta_W, \Delta_T, E_T)$. Unfortunately, this polynomial is not unique, since $\ker(\vartheta(R)) \neq 0$. By Theorem 3.3, for $f \in M_{k,0}(\Gamma_0(T))_R$, there exists a unique polynomial $\varphi_f(U, V) \in R[U, V]$ such that $f = \varphi_f(\Delta_W, \Delta_T)$. Using $\varphi_f(U, V)$ and Theorem 3.5, we study the properties of the weight filtration for $M_{k,l}(\Gamma_0(T))_{A_{\mathfrak{p}}}$ (cf. Theorem 5.2). This can be thought of as a generalization of the work in [Vin10] from $\mathrm{GL}_2(A)$ to $\Gamma_0(T)$.

3.5. **Application:** In the last section, as an application of the results of this article, we shall prove a result on mod-$\mathfrak{p}$ congruences for Drinfeld modular forms of level $\Gamma_0(\mathfrak{p}T)$ for $\mathfrak{p} \neq (T)$, which is a variant of [DK1, Theorem 1.5].

Since the methods of this article work for any degree one prime, it is of interest to study similar results for the remaining genus zero case i.e., for $\deg(\mathfrak{n}) = 2$, and also for higher genus.

4. Structure of Drinfeld modular forms for $\Gamma_0(T)$

In this section, first we introduce two important Drinfeld modular forms and study their properties. Recall that 0 and ∞ are the only cusps of the Drinfeld modular curve $X_0(T) := \overline{\Gamma_0(T)\backslash\Omega}$ and the operator $W_T := \left(\begin{smallmatrix} 0 & -1 \\ T & 0 \end{smallmatrix}\right)$ permutes the cusps. Consider the functions

$$\Delta_T(z) := \frac{g_1(Tz) - g_1(z)}{T^q - T} \text{ and } \Delta_W(z) := \frac{T^q g_1(Tz) - T g_1(z)}{T^q - T}. \qquad (4.1)$$

An easy check implies that $\Delta_T, \Delta_W \in M_{q-1,0}(\Gamma_0(T))$. By [Cho09, Page 3], the u-expansion of Δ_T at ∞ is given by

$$\Delta_T = u^{q-1} - u^{q(q-1)} + \cdots \in A[[u]].$$

Since $g_1(z) = 1 - (T^q - T)u^{q-1} - (T^q - T)u^{(q^2-q+1)(q-1)} + \cdots$ and $u(Tz) = u^q + \cdots$, the u-expansion of Δ_W at ∞ is given by

$$\Delta_W = 1 + Tu^{q-1} - T^q u^{q(q-1)} + \cdots \in A[[u]].$$

Next, we study some important properties of Δ_T, Δ_W, and E_T. Before, we state them as a proposition, first let us recall the dimension formula for $\Gamma_0(T)$.

Proposition 4.1. *The C-dimension of $M_{k,l}(\Gamma_0(T))$ is $1 + r_{k,l}$, where $r_{k,l} := \frac{k-2l}{q-1}$.*

Proof. If $l \equiv 0 \pmod{q-1}$, then the proposition follows from [Gek86, Proposition 6.4]. If $l \not\equiv 0 \pmod{q-1}$, then the proposition can be deduced from the discussion in [BV19, §4.3], since any Drinfeld modular form of non-trivial type is cuspidal. Now, we follow the notations as in [BV19].

Let $\mathcal{B}_k^1(\Gamma_1(T)) := \{c_j : 0 \le j \le k-2\}$ be a basis of $M_k^1(\Gamma_1(T))$ as in [BV19, §4.2] and $C_j := \{c_i : i \equiv j \pmod{q-1}\}$. From the discussion in [BV19, §4.3], we get $M_{k,l}(\Gamma_0(T)) = M_{k,l}^1(\Gamma_0(T)) = \text{span}\{c_i : c_i \in C_{l-1}\}$. Hence $\dim M_{k,l}(\Gamma_0(T)) = |C_{l-1}| = n = 1 + r_{k,l}$. One can also use Riemann-Roch Theorem to give an alternate proof. $\qquad\square$

We now define a Victor Miller basis for $M_{k,l}(\Gamma_0(\mathfrak{n}))$. For any $f \in M_{k,l}(\Gamma_0(\mathfrak{n}))$, suppose the u-expansion of f at ∞ is given by $\sum_{i \ge 0} a_f(i(q-1) + l)u^{i(q-1)+l}$. Then, we define $b_f(i) := a_f(i(q-1) + l)$.

Definition 4.2 (Victor Miller basis). We say that a basis $f_1, f_2, \ldots, f_r$ of $M_{k,l}(\Gamma_0(\mathfrak{n}))$, where $r = \dim_C M_{k,l}(\Gamma_0(\mathfrak{n}))$, is a Victor Miller basis for $M_{k,l}(\Gamma_0(\mathfrak{n}))$ if $b_{f_j}(i) = \delta_{ij}$ for all $1 \le i, j \le r$.

We now state some important properties of Δ_T, Δ_W, and E_T.

Proposition 4.3. *Let Δ_W, Δ_T, E_T be as in (4.1) and (2.1). Then, we have*

(1) $g_1 = \Delta_W - T^q \Delta_T$.

(2) *The modular form Δ_T(resp., Δ_W) vanishes $q-1$ times at ∞ (resp., at 0) and non-zero on $\Omega \cup \{0\}$ (resp., on $\Omega \cup \{\infty\}$). Hence, the modular forms Δ_T, Δ_W are algebraically independent.*

(3) *The set $S := \{\Delta_W^{r_{k,l}} E_T^l, \Delta_W^{r_{k,l}-1} \Delta_T E_T^l, \ldots, \Delta_W \Delta_T^{r_{k,l}-1} E_T^l, \Delta_T^{r_{k,l}} E_T^l\}$ is a basis for $M_{k,l}(\Gamma_0(T))$. In particular, the space $M_{k,l}(\Gamma_0(T))$ has an integral basis, i.e., a basis consisting of modular forms with coefficient in A.*

(4) *There exists a Victor Miller basis for the C-vector space $M_{k,l}(\Gamma_0(T))$.*

(5) *For $k \in \mathbb{N} \cup \{0\}$, the map $\eta : M_{k-2l,0}(\Gamma_0(T)) \to M_{k,l}(\Gamma_0(T))$ defined by $f \to fE_T^l$ is an isomorphism.*

(6) *We have $E_T^{q-1} = \Delta_W \Delta_T$. In particular, the function $E_T \in M_{2,1}(\Gamma_0(T))$ vanishes exactly once at the cusps $0, \infty$ and non-vanishing elsewhere.*

(7) $\partial \Delta_W = -\Delta_W E_T$, $\partial \Delta_T = 0$, $\partial(E_T) = E_T^2$, *where ∂f as in Definition 6.2.*

(8) $h(z) = \partial(g_1(z)) = \partial(g_1(Tz)) = -\Delta_W(z)E_T(z)$, $\Delta = -\Delta_W^q \Delta_T$.

In literature, there is a folklore conjecture that there exists an integral basis for $M_{k,l}(\Gamma_0(\mathfrak{n}))$. The known results for $M_{k,l}(\text{GL}_2(A))$, $M_{k,0}(\Gamma_0(T))$ are due to Gekeler ([Gek88]), Cornelissen ([Cor97, Proposition 4.6.2]) respectively. For general level $\mathfrak{n}$, the best known result is for $M_{2,1}^2(\Gamma_0(\mathfrak{n}))$ due to Armana ([Arm08, Section 4.2]). Note that, Proposition 4.3(3) implies that the space $M_{k,l}(\Gamma_0(T))$ has an integral basis.

Proof of Proposition 4.3. (1) This follows from the definitions of Δ_W and Δ_T.

(2) By u-expansion of Δ_T at ∞, we see that Δ_T vanishes exactly $q-1$ times at ∞. By [Cho09, Proposition 3.3], the function Δ_T is non-zero on $\Omega \cup \{0\}$. Since $\Delta_W = -T^{\frac{q+1}{2}} \Delta_T \mid_{q-1,0} W_T$ and W_T permutes the cusps, we see that Δ_W vanishes $q-1$ times at 0 and non-zero on $\Omega \cup \{\infty\}$.

We now show that the functions Δ_T, Δ_W are algebraically independent. Suppose there is an algebraic relation between Δ_T and Δ_W. The graded structure of $M(\Gamma_0(T))$ implies that there exists an algebraic relation of minimal weight k and of the form

$$c_0 \Delta_W^{\frac{k}{q-1}} + \sum_{i=1}^{n-1} c_i \Delta_W^{x_i} \Delta_T^{y_i} + c_n \Delta_T^{\frac{k}{q-1}} = 0, \tag{4.2}$$

with $(x_i + y_i)(q-1) = k$, $x_i y_i \neq 0$ for all $1 \leq i \leq n-1$, and $c_0 c_n \neq 0$. Evaluating both sides of (4.2) at ∞, we get $c_0 = 0$. This contradicts the minimality of k.

(3) By Part (2), the set S is linearly independent. Since $\#S = \dim M_{k,l}(\Gamma_0(T))$, S is a basis for the C-vector space $M_{k,l}(\Gamma_0(T))$. Since the elements of S have coefficients in A, S is an integral basis for $M_{k,l}(\Gamma_0(T))$.

(4) Let S be as in Part (3). For $0 \leq j \leq r_{k,l}$, the u-expansion of elements of S is given by

$$f_j := \Delta_W^{r_{k,l}-j} \Delta_T^j E_T^l = u^{j(q-1)+l} + \cdots \in A[[u]].$$

Then, $B := [b_{f_j}(i)]_{0 \leq j,i \leq r_{k,l}}$ is an upper triangular matrix with diagonal entries 1. Since the row reduced Echelon form of B is an identity matrix, the existence of a Victor Miller basis for $M_{k,l}(\Gamma_0(T))$ follows.

(5) By Part (3), the map η is surjective. By Proposition 4.1, the domain, co-domain have same dimension, hence η is an isomorphism.

(6) By [AW20, Theorem 6.3], it is enough to show that the coefficients of u^0, u^{q-1} and $u^{2(q-1)}$ are equal in the u-expansions of E_T^{q-1} and $\Delta_W \Delta_T$ at ∞. This is an easy check. The latter part follows from the vanishing (resp., non-vanishing) of Δ_T, Δ_W on $\Omega \cup \{0, \infty\}$.

(7) By [AW20, Theorem 6.3], it is enough to show that the coefficients of u and u^q are equal in the u-expansions of $\partial \Delta_T$ and 0 (resp., $\partial \Delta_W$ and $\Delta_W E_T$) at ∞. The third equality can be verified by showing that the coefficients of u^2 are equal in the u-expansions of $\partial(E_T)$ and E_T^2. This is an easy check.

(8) By [AW20, Theorem 6.3], it is enough to show that the coefficients of u and u^q are equal in the u-expansions of $h(z), \partial(g_1(z)), \partial(g_1(Tz))$ and $\Delta_W(z) E_T(z)$ at ∞. This is an easy check. The last relation follows from Part(6) and $h^{q-1} = -\Delta$.

$\square$

4.1. **On the structure of $M(\Gamma_0(T))_R$:** We are now ready to prove one of the main results of this article, which is a generalization of [Gek88, Theorem 5.13] from $GL_2(A)$ to $\Gamma_0(T)$.

Theorem 4.4. *The R-algebra $M(\Gamma_0(T))_R$ is generated by Δ_W, Δ_T and E_T. Moreover, the surjective map $\vartheta(R) : R[U, V, Z] \twoheadrightarrow M(\Gamma_0(T))_R$ defined by $U \to$*

$\Delta_W, V \to \Delta_T, Z \to E_T$ *induces an isomorphism*

$$R[U, V, Z]/(UV - Z^{q-1}) \cong M(\Gamma_0(T))_R. \tag{4.3}$$

To prove Theorem 4.4, first we need to introduce the notion of weight, type for a polynomial in $R[U, V, Z]$.

Definition 4.5. Let $k \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{Z}/(q-1)\mathbb{Z}$. A polynomial $\varphi(U, V, Z) = \sum_{i=1}^n c_i U^{x_i} V^{y_i} Z^{l_i} \in R[U, V, Z]$ is said to be of weight k, type l if
  (1) $(x_i + y_i)(q-1) + 2l_i = k, \forall 1 \le i \le n$ and
  (2) $l_1 \equiv l_2 \equiv \cdots \equiv l_n \equiv l \pmod{q-1}$.

Remark 4.6. Definition 4.5 is also valid for polynomials over $\mathbb{F}_{\mathfrak{p}}$. By definition, the variables U, V are of weight $q-1$, type 0 and Z is of weight 2, type 1.

Let $R[U, V, Z]_{k,l}$ denote the set of polynomials of weight k, type l. Then

$$R[U, V, Z] = \bigoplus_{\substack{k \in \mathbb{N} \cup \{0\} \\ l \in \mathbb{Z}/(q-1)\mathbb{Z}}} R[U, V, Z]_{k,l}. \tag{4.4}$$

Proof of Theorem 4.4. For simplicity, we denote $\vartheta(R)$ by ϑ. Let $f \in M_{k,l}(\Gamma_0(T))_R$. Since $B := \{\Delta_W^{r_{k,l}-j} \Delta_T^j E_T^l | 0 \le j \le r_{k,l}\}$ is a basis for the C-vector space $M_{k,l}(\Gamma_0(T))$ (cf. Proposition 4.3 (3)), f can be written as

$$f = c_0 \Delta_W^{r_{k,l}} E_T^l + c_1 \Delta_W^{r_{k,l}-1} \Delta_T E_T^l + \cdots + c_{r_{k,l}} \Delta_T^{r_{k,l}} E_T^l \text{ with } c_j \in C. \tag{4.5}$$

For $0 \le j \le r_{k,l}$, the u-expansion of elements of B is given by

$$\Delta_W^{r_{k,l}-j} \Delta_T^j E_T^l = u^{j(q-1)+l} + \cdots.$$

Comparing the coefficients in u-expansions on both sides of (4.5) we get $c_j \in R$ for all $j \in \{0, \ldots, r_{k,l}\}$. This shows that the map ϑ is surjective. In particular, this implies that the R-algebra $M(\Gamma_0(T))_R$ is generated by Δ_W, Δ_T and E_T.

It remains to show that $\ker \vartheta = (UV - Z^{q-1})$. By Proposition 4.3(6), we get $(UV - Z^{q-1}) \subseteq \ker \vartheta$. To show the equality, by (4.4), it suffices to show that for any $f \in R[U, V, Z]_{k,l}$ such that $\vartheta(f) = 0$ then $f \in (UV - Z^{q-1})$. Recall that $0 \le l \le q-2$ is a lift of $l \in \mathbb{Z}/(q-1)\mathbb{Z}$.

Let $f = \sum_{i=0}^n c_i U^{x_i} V^{y_i} Z^{l_i} \in R[U, V, Z]_{k,l}$ be a polynomial of weight k, type l such that $\vartheta(f) = 0$. The condition (2) in Definition 4.5 implies that f can be written as $f = \sum_{i=0}^n c_i U^{x_i} V^{y_i} Z^{l+m_i(q-1)}$ for some $m_i \in \mathbb{Z}$. By Proposition 4.3(6), we have

$$\vartheta(f) = E_T^l \cdot \sum_{i=0}^n c_i \Delta_W^{x_i+m_i} \Delta_T^{y_i+m_i} = 0. \tag{4.6}$$

Define $S := \{y_i + m_i \mid 0 \le i \le n\}$, $T := \{x_i + m_i \mid 0 \le i \le n\}$. For $(a,b) \in S \times T$, we define

$$I_{(a,b)} := \{i \mid y_i + m_i = a, \ x_i + m_i = b\}. \tag{4.7}$$

The expression in (4.6) is reduced to $E_T^l \cdot \sum_{(a,b) \in S \times T} c_{a,b} \Delta_W^b \Delta_T^a = 0$, where $c_{a,b} = \sum_{i \in I_{(a,b)}} c_i$. Since Δ_W, Δ_T are algebraically independent (cf. Proposition 4.3 (2)) we get

$$c_{a,b} = \sum_{i \in I_{(a,b)}} c_i = 0 \quad \text{for } (a,b) \in S \times T. \tag{4.8}$$

Let $f_{a,b} := \sum_{i \in I_{(a,b)}} c_i U^{x_i} V^{y_i} Z^{l+m_i(q-1)}$ for $(a,b) \in S \times T$. Since $f = \sum_{(a,b) \in S \times T} f_{a,b}$, it is enough to show that $f_{a,b} \in (UV - Z^{q-1})$.

Let $m_r = \inf_{i \in I_{(a,b)}} m_i$. By (4.8), the expression for $f_{a,b}$ becomes

$$\sum_{\substack{i \in I_{(a,b)} \\ i \neq r}} c_i U^{x_i} V^{y_i} Z^{l+m_i(q-1)} - \Big(\sum_{\substack{i \in I_{(a,b)} \\ i \neq r}} c_i \Big) U^{x_r} V^{y_r} Z^{l+m_r(q-1)}. \qquad (4.9)$$

For simplicity, we let $x(j;i) := x_j - x_i$, $y(j;i) := y_j - y_i$, and $m(i;j) := m_i - m_j$. Then, the expression in (4.9) reduced to

$$Z^l \sum_{\substack{i \in I_{(a,b)} \\ i \neq r}} c_i Z^{m_r(q-1)} U^{x_i} V^{y_i} \Big(Z^{(q-1)m(i;r)} - U^{x(r;i)} V^{y(r;i)} \Big)$$

By (4.7), we see that $x(r;i) = y(r;i) = m(i;r)$ for $i \in I_{(a,b)}$. Hence, the expression in the last bracket becomes $Z^{(q-1)m(i;r)} - (UV)^{m(i;r)}$, which is a multiple of $Z^{q-1} - UV$. This implies $f_{a,b} \in (UV - Z^{q-1})$ for all $(a,b) \in S \times T$ and this proves the claim. $\qquad \square$

By Theorem 4.4, for $f \in M_{k,l}(\Gamma_0(T))_R$, there exists a polynomial $\eta(U,V,Z) \in R[U,V,Z]$ such that $f = \eta(\Delta_W, \Delta_T, E_T)$. Unfortunately, the polynomial η is not unique since $\ker \vartheta(R) \neq 0$.

In the next section, we show that for each $f \in M_{k,l}(\Gamma_0(T))_R$ there exists a unique polynomial $\varphi_f(U,V) \in R[U,V]$ such that $f = \varphi_f(\Delta_W, \Delta_T)E_T^l$. To do this, we need to describe the structure of the R-algebra $M^0(\Gamma_0(T))_R$.

4.2. **On the structure of $M^0(\Gamma_0(T))_R$:** As a consequence of Proposition 4.3, we produce an explicit set of generators for the R-algebra $M^0(\Gamma_0(T))_R$.

Theorem 4.7. *The map*

$$\vartheta_0(R) : R[U,V] \longrightarrow M^0(\Gamma_0(T))_R$$

defined by $(U,V) \to (\Delta_W, \Delta_T)$ is an isomorphism. In particular, the R-algebra $M^0(\Gamma_0(T))_R$ is generated by Δ_W, Δ_T, i. e., $M^0(\Gamma_0(T))_R = R[\Delta_W, \Delta_T]$.

Proof. The surjectivity of $\vartheta_0(R)$ is clear from (4.5) by choosing $l = 0$. The injectivity of $\vartheta_0(R)$ follows from Proposition 4.3(2). $\qquad \square$

We now introduce the notion of isobaric polynomial in $R[U,V]$.

Definition 4.8. A polynomial $\varphi(U,V) = \sum_{i=1}^n c_i U^{x_i} V^{y_i} \in R[U,V]$ is said to be an isobaric polynomial of weight $k \in \mathbb{N} \cup \{0\}$ if $(x_i + y_i)(q-1) = k$ for all $1 \leq i \leq n$.

Remark 4.9. Definition 4.8 is also valid for polynomials over $\mathbb{F}_{\mathfrak{p}}$. By definition, the variables U, V are isobaric polynomials of weight $q - 1$.

By Theorem 4.7, for $f \in M_{k,0}(\Gamma_0(T))_R$, there exists a unique polynomial $\varphi_f(U,V) = \sum_{i=1}^n c_i U^{x_i} V^{y_i} \in R[U,V]$ such that $f = \varphi_f(\Delta_W, \Delta_T) = \sum_{i=1}^n c_i \Delta_W^{x_i} \Delta_T^{y_i}$. Since the weight of f is k, we have $(x_i + y_i)(q-1) = k$ for all $1 \leq i \leq n$. Hence the polynomial $\varphi_f(U,V)$ is an isobaric polynomial of weight k. These polynomials play an important role in rest of the article.

The novelty of our work is that we use Theorem 4.7 to show that for each $f \in M_{k,l}(\Gamma_0(T))_R$ there exists a unique polynomial $\varphi_f(U, V) \in R[U, V]$ such that $f = \varphi_f(\Delta_W, \Delta_T)E_T^l$. This is done as follows.

To each $f \in M_{k,l}(\Gamma_0(T))_R$, we define $c_T(f) := \frac{f}{E_T^l} \in M_{k-2l,0}(\Gamma_0(T))$ (cf. Proposition 4.3(5)). Since the u-expansion of E_T^l at ∞ is $u^l(1 + \cdots) \in A[[u]]$, an easy verification shows that $c_T(f) \in M_{k-2l,0}(\Gamma_0(T))_R$. By Theorem 4.7, there exists a unique isobaric polynomial $\varphi_{c_T(f)}(U, V) \in R[U, V]$ of weight $k - 2l$ such that $c_T(f) = \varphi_{c_T(f)}(\Delta_W, \Delta_T)$. This implies that $f = \varphi_{c_T(f)}(\Delta_W, \Delta_T)E_T^l$. For simplicity, we denote $\varphi_{c_T(f)}(U, V)$ by $\varphi_f(U, V)$. We summarize the discussion above as a proposition.

Proposition 4.10. *For $f \in M_{k,l}(\Gamma_0(T))_R$, there exists a unique isobaric polynomial $\varphi_f(U, V) \in R[U, V]$ such that $c_T(f) = \varphi_f(\Delta_W, \Delta_T)$. Hence, $f = \varphi_f(\Delta_W, \Delta_T)E_T^l$.*

We now record a useful lemma.

Lemma 4.11. *For $i = 1, 2$, if $f_i \in M_{k_i,l_i}(\Gamma_0(T))$ such that at least one of f_1, f_2 is non-zero modulo $\mathfrak{p}$. Then, $f_1 \equiv f_2 \pmod{\mathfrak{p}}$ if and only if $l_1 \equiv l_2 \pmod{q-1}$ and $c_T(f_1) \equiv c_T(f_2) \pmod{\mathfrak{p}}$.*

Proof. Let $0 \le l \le q - 2$ be an integer such that $l_1 \equiv l_2 \equiv l \pmod{q-1}$ and $c_T(f_1) \equiv c_T(f_2) \pmod{\mathfrak{p}}$. Since E_T^l has coefficients in A, the congruence $c_T(f_1) \equiv c_T(f_2) \pmod{\mathfrak{p}}$ gives $c_T(f_1)E_T^l \equiv c_T(f_2)E_T^l \pmod{\mathfrak{p}}$, i.e., $f_1 \equiv f_2 \pmod{\mathfrak{p}}$.

Conversely, we suppose $f_1 \equiv f_2 \pmod{\mathfrak{p}}$. The u-expansions of f_1 and f_2 imply that $l_1 \equiv l_2 \equiv l \pmod{q-1}$ for $0 \le l \le q - 2$. Since $f_i = c_T(f_i)E_T^l$, one gets $c_T(f_1)E_T^l \equiv c_T(f_2)E_T^l \pmod{\mathfrak{p}}$. The u-expansion of E_T^l at ∞ is $u^l(1 + \cdots) \in A[[u]]$, so we obtain $c_T(f) \equiv c_T(g) \pmod{\mathfrak{p}}$. $\square$

By Lemma 4.11, the congruence between Drinfeld modular forms of arbitrary type reduces to the congruence between Drinfeld modular forms of type 0. Thus, the study of the weight filtration heavily depends on the structure of mod-$\mathfrak{p}$ reduction of $M_{\mathfrak{p}}^0(\Gamma_0(T))$, which is the content of the following section.

4.3. **On the structure of mod-$\mathfrak{p}$ reduction of $M_{\mathfrak{p}}^0(\Gamma_0(T))$:** We now describe the structure of $M_{\mathfrak{p}}^0(\Gamma_0(T))$ under mod $\mathfrak{p}$-reduction with $\mathfrak{p} \ne (T)$.

Definition 4.12. Let $M_{\mathfrak{p}}^0(\Gamma_0(T))$ denote the ring of Drinfeld modular forms for $\Gamma_0(T)$ of type 0 with $\mathfrak{p}$-integral u-expansion at ∞. We define

$$\overline{M_{\mathfrak{p}}^0}(\Gamma_0(T)) := \{\overline{f} \in \mathbb{F}_{\mathfrak{p}}[[u]] \mid \exists f \in M_{\mathfrak{p}}^0(\Gamma_0(T)) \text{ such that } f \equiv \overline{f} \pmod{\mathfrak{p}}\}.$$

Let $\phi_d(U, V)$ be the unique isobaric polynomial of weight $q^d - 1$ attached to $g_d \in M_{q^d-1,0}(\Gamma_0(T))$ such that $g_d = \phi_d(\Delta_W, \Delta_T)$. Let $\overline{\phi}_d(U, V) := \phi_d(U, V) \pmod{\mathfrak{p}}$.

Theorem 4.13. *The surjective map $\iota : \mathbb{F}_{\mathfrak{p}}[U, V] \twoheadrightarrow \overline{M_{\mathfrak{p}}^0}(\Gamma_0(T))$ defined by $U \to \overline{\Delta}_W$ and $V \to \overline{\Delta}_T$ induces an isomorphism*

$$\mathbb{F}_{\mathfrak{p}}[U, V]/(\overline{\phi}_d(U, V) - 1) \cong \overline{M_{\mathfrak{p}}^0}(\Gamma_0(T)).$$

To prove Theorem 4.13, we study the behavior of isobaric polynomials under a certain change of variables and we show that $\overline{\phi}_d(U, V)$ is square-free.

Let $B_d(X, Y_1)$ (resp., $A_d(X, Y)$) be the unique isobaric polynomial attached to $g_d \in M_{q^d-1,0}(\mathrm{GL}_2(A))$ such that $g_d = B_d(g_1, \Delta)$ (resp., $g_d = A_d(g_1, h)$). The relation between Δ, h implies $Y_1 = -Y^{q-1}$. By Proposition 4.3, we get $\phi_d(U, V) = B_d(U - T^q V, -U^q V)$. By [Gek88, Proposition 6.9], the relation among g_d gives

$$\phi_d(U, V) = \phi_{d-1}(U, V)(U - T^q V)^{q^{d-1}} + (T^{q^{d-1}} - T)\phi_{d-2}(U, V)(U^q V)^{q^{d-2}}, \quad (4.10)$$

where $\phi_0(U, V) = 1$, $\phi_1(U, V) = U - T^q V$.

Remark 4.14. From (4.10), we see that $U \mid \phi_d(U, V)$ if and only if $U \mid \phi_{d-1}(U, V)$. Since $\phi_1(U, V) = U - T^q V$, we get $U \nmid \phi_d(U, V)$ for all d. Similarly, we get $U \nmid \overline{\phi}_d(U, V)$ for all d. These properties do hold for the variable V.

Consider the Tate-Drinfeld module TD over $K((u))$ defined by

$$\mathrm{TD}_T = T\tau^0 + g_1(u)\tau + \Delta(u)\tau^2.$$

Let $\mathrm{TD}_\pi = \sum_{0 \le i \le 2d} l_i \tau^i$ and $F_i(X, Y_1) \in A[X, Y_1]$ be the unique isobaric polynomial attached to $\overline{l}_i$ such that $F_i(g_1, \Delta) = l_i$. Define

$$f_i(U, V) := F_i(U - T^q V, -U^q V).$$

By [Gek88, Corollary 12.3], we have $\overline{\phi}_d(U, V) = \overline{f}_d(U, V)$. We now show that the polynomial $\overline{\phi}_d(U, V)$ is square-free. The proof of this is analogous to the proof of [Gek88, Proposition 11.7].

Proposition 4.15. *The polynomial $\overline{\phi}_d(U, V)$ is square-free.*

Proof. Let $H(U) := \overline{f}_{2d-1}(U, 1)$. By [Gek88, Lemma 11.6], we get $H'(U) = (-U)^a$ for some $a \in \mathbb{N}$. The possible multiple factors of $\overline{f}_{2d-1}(U, V)$ are U or V. Since $\overline{f}_d \mid \overline{f}_{2d-1}$, the possible multiple factors of $\overline{f}_d(U, V)$ are U or V. This is a contradiction, since $\overline{\phi}_d(U, V)$ is neither divisible by U nor by V (cf. Remark 4.14). □

Remark 4.16. If $\mathfrak{p} = (T)$, then $\overline{\phi}_1(U, V) = U$. So, $U \mid \overline{\phi}_1(U, V)$. To avoid this situation, we work with a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \ne (T)$.

We have now all the information that is required to prove Theorem 4.13.

Proof of Theorem 4.13. By Theorem 4.7, ι is surjective. Clearly, $\ker(\iota)$ is a non-maximal prime ideal and $(\overline{\phi}_d(U, V) - 1) \subseteq \ker(\iota)$. Since the Krull dimension of $\mathbb{F}_\mathfrak{p}[U, V]$ is 2, it suffices to show that $\overline{\phi}_d(U, V) - 1$ is irreducible.

Suppose $\overline{\phi}_d(U, V) - 1$ is reducible. Let $\overline{\phi}_d(U, V) - 1 = RS$, where $R = \sum_{i \le m} R_i$ and $S = \sum_{j \le n} S_j$ where R_i, S_j denote isobaric polynomials of weight i, j respectively. By comparing the highest, lowest weights, we get $\overline{\phi}_d(U, V) = R_m S_n$, $R_0 S_0 = -1$, respectively. Since $\overline{\phi}_d(U, V)$ is square-free, the polynomials R_m, S_n have no common factor. Again by comparing the other weights, we obtain $R_m S_{n-1} + R_{m-1} S_n = 0$. Since R_m, S_n have no common factor, we conclude that $R_{m-1} = 0, S_{n-1} = 0$. By induction, we get $R_i = 0$ for $i < m$ and $S_j = 0$ for $j < n$. This implies that $R_0 S_0 = 0$, which is a contradiction. This proves the claim. □

We have a corollary.

Corollary 4.17. *For $i = 1, 2$, let $f_i \in M_{k_i,0}(\Gamma_0(T))$ be a Drinfeld modular form with $\mathfrak{p}$-integral u-expansion at ∞ such that $f_i \not\equiv 0 \pmod{\mathfrak{p}}$. If $f_1 \equiv f_2 \pmod{\mathfrak{p}}$ then $k_1 \equiv k_2 \pmod{q^d - 1}$.*

Proof. Let $k_1 > k_2$. For $i = 1, 2$, let $\varphi_{f_i}(U, V)$ be the unique isobaric polynomial attached to f_i. By Theorem 4.13, the condition $f_1 \equiv f_2 \pmod{\mathfrak{p}}$ implies that $\overline{\varphi}_{f_1}(U, V) - \overline{\varphi}_{f_2}(U, V) \in (\overline{\phi}_d(U, V) - 1)$. Thus $\overline{\varphi}_{f_1}(U, V) - \overline{\varphi}_{f_2}(U, V) = R(\overline{\phi}_d(U, V) - 1)$ for some $R = \sum_{i=0}^{n} R_{m_i} \in \mathbb{F}_{\mathfrak{p}}[U, V]$, where each R_{m_i} is a non-zero isobaric polynomial of weight m_i with $m_i < m_{i+1}$. By comparing the terms of same weights on both sides, we get

$$\overline{\varphi}_{f_1}(U, V) = R_{m_n}\overline{\phi}_d(U, V). \tag{4.11}$$

This implies $\overline{\phi}_d(U, V) \mid \overline{\varphi}_{f_1}(U, V)$. Comparing the remaining weights, we get

$$\overline{\varphi}_{f_2}(U, V) = R_{m_0} \text{ and } R_{m_i}\overline{\phi}_d(U, V) = R_{m_{i+1}} \text{ for } 0 \leq i \leq n - 1. \tag{4.12}$$

Equating the weights in (4.11) and (4.12), we get

$$m_n = k_1 - (q^d - 1), \; m_{i+1} = m_i + (q^d - 1) \text{ for } 0 \leq i \leq n - 1, \text{ and } m_0 = k_2.$$

Now, it is easy to see that $(q^d - 1) | (k_1 - k_2)$. This proves the corollary. $\square$

A more general version of the above corollary can be found in the work of Hattori (cf. [Hat20, Theorem 4.16]).

5. WEIGHT FILTRATION FOR $f \in M_{k,l}(\Gamma_0(T))$

In this section, we define the weight filtration for $f \in M_{k,l}(\Gamma_0(T))$ and study its properties. Let $M_k(\Gamma_0(T))$ denote the space of Drinfeld modular forms of weight k (any type) for $\Gamma_0(T)$.

Definition 5.1. Let $f \in M_{k,l}(\Gamma_0(T))$ be a Drinfeld modular form with $\mathfrak{p}$-integral u-expansion at ∞ such that $f \not\equiv 0 \pmod{\mathfrak{p}}$. We define the weight filtration of f, which we denote by $w(\overline{f})$, as

$$w(\overline{f}) := \inf\{k_0 | \exists \; g \in M_{k_0}(\Gamma_0(T)) \text{ with } g \equiv f \pmod{\mathfrak{p}}\}.$$

By Lemma 4.11, we get $g \in M_{k_0,l}(\Gamma_0(T))$. If $f \equiv 0 \pmod{\mathfrak{p}}$, then $w(\overline{f}) := -\infty$.

In the following theorem, we show that the properties of the weight filtration of $f \in M_{k,l}(\Gamma_0(T))$ can be deduced from the relevant properties of $c_T(f)$.

Theorem 5.2. *Let $f \in M_{k,l}(\Gamma_0(T))$ be a Drinfeld modular form with $\mathfrak{p}$-integral u-expansion at ∞ and $f \not\equiv 0 \pmod{\mathfrak{p}}$. Let $\varphi_f(U, V)$ be the unique isobaric polynomial attached to $c_T(f)$. Then*

    (1) $w(\overline{f}) = w(\overline{c_T(f)}) + 2l$,
    (2) $w(\overline{f}) \equiv k \pmod{q^d - 1}$,
    (3) $w(\overline{f}) < k$ *if and only if* $\overline{\phi}_d(U, V) \mid \overline{\varphi}_f(U, V)$.

Proof. Recall that l denotes a representative of $l \in \mathbb{Z}/(q-1)\mathbb{Z}$ with $0 \leq l \leq q-2$. The proof of the theorem is divided into two parts, i. e., $k = 2l$ and $k > 2l$ (note that $\dim M_{k,l}(\Gamma_0(T)) = 0$ if $k < 2l$).

If $k = 2l$, then $f = c \cdot E_T^l$ with $c \not\equiv 0 \pmod{\mathfrak{p}}$. We show that $w(\overline{E_T^l}) = 2l$. If $w(\overline{E_T^l}) = e < 2l$, then $E_T^l \equiv g \pmod{\mathfrak{p}}$ for some $g \in M_{e,l}(\Gamma_0(T))$. Since $e < 2l$, we get that $g = 0$ (cf. Proposition 4.1). This is a contradiction. This implies part(1), part(2) trivially, and part(3) is vacuously true.

We now assume $k > 2l$. If $k' = w(\overline{f})$ then there exists $g \in M_{k',l}(\Gamma_0(T))$ such that $f \equiv g \pmod{\mathfrak{p}}$. By definition, $c_T(f) \in M_{k-2l,0}(\Gamma_0(T))$, $c_T(g) \in M_{k'-2l,0}(\Gamma_0(T))$. By Lemma 4.11, the congruence $f \equiv g \pmod{\mathfrak{p}}$ implies that $c_T(f) \equiv c_T(g) \pmod{\mathfrak{p}}$. Thus, $w(\overline{c_T(f)}) \leq k' - 2l$.

We now show that $w(\overline{c_T(f)}) = k' - 2l$. If $w(\overline{c_T(f)}) < k' - 2l$, then there exists $f_1 \in M_{w(\overline{c_T(f)}),0}(\Gamma_0(T))$ such that $c_T(f) \equiv f_1 \pmod{\mathfrak{p}}$. This implies that $f \equiv f_1 E_T^l \pmod{\mathfrak{p}}$. Consequently, $w(\overline{f}) \leq w(\overline{c_T(f)}) + 2l < k' = w(\overline{f})$, which is a contradiction. Hence, we have $w(\overline{c_T(f)}) = w(\overline{f}) - 2l$. This proves part(1) of the theorem.

By Corollary 4.17 we have $w(\overline{c_T(f)}) \equiv k - 2l \pmod{q^d - 1}$. Now, part(2) follows from part(1).

Now, we prove part(3). If $w(\overline{f}) < k$, then there exists $g \in M_{w(\overline{f}),l}(\Gamma_0(T))$ such that $f \equiv g \pmod{\mathfrak{p}}$ and hence $c_T(f) \equiv c_T(g) \pmod{\mathfrak{p}}$ (cf. Lemma 4.11). The equation (4.11) in the proof of Corollary 4.17 implies that $\overline{\phi}_d(U,V) \mid \overline{\varphi}_f(U,V)$. This proves the forward implication.

For the reverse implication, if $\overline{\phi}_d(U,V) \mid \overline{\varphi}_f(U,V)$ then there exists $\overline{\psi}(U,V) \in \mathbb{F}_{\mathfrak{p}}[U,V]$ such that $\overline{\varphi}_f(U,V) = \overline{\phi}_d(U,V)\overline{\psi}(U,V)$. Since $\overline{\varphi}_f(U,V)$, $\overline{\phi}_d(U,V)$ are isobaric, $\overline{\psi}(U,V)$ is also isobaric with weight $(k - 2l) - (q^d - 1)$. Let $\psi(U,V) \in A_{\mathfrak{p}}[U,V]$ be an isobaric lift of $\overline{\psi}(U,V)$ with same weight. Then $g := \psi(\Delta_W, \Delta_T)$ is a Drinfeld modular form of weight $(k - 2l) - (q^d - 1)$. Since $\overline{\varphi}_f(U,V) = \overline{\phi}_d(U,V)\overline{\psi}(U,V)$, we get that $\overline{\varphi}_f(\Delta_W, \Delta_T) = \overline{\phi}_d(\Delta_W, \Delta_T)\overline{\psi}(\Delta_W, \Delta_T)$. The congruence $g_d \equiv 1 \pmod{\mathfrak{p}}$ implies that $c_T(f) \equiv g \pmod{\mathfrak{p}}$ and $f \equiv gE_T^l \pmod{\mathfrak{p}}$. Hence, we have $w(\overline{f}) < k$. $\qquad\square$

6. An application to mod-$\mathfrak{p}$ congruences for $\mathfrak{p} \neq (T)$

In this section, as an application of Theorem 4.7 and Theorem 5.2, we prove a certain relation between mod-$\mathfrak{p}$ congruences for Drinfeld modular forms f, g of level $\Gamma_0(\mathfrak{p}T)$ and signs of the Fricke involution on f, g.

For classical modular forms, Calegari and Stein (cf. [CS04, Conjecture 4]) conjectured that the eigenvalues of Fricke involution on f and g have opposite signs if there is a mod p congruence between the cusp form g of weight 4 and derivative of the cusp form f of weight 2 for $\Gamma_0(p)$. In [DK1], we gave a proof of this conjecture for Drinfeld modular forms f of weight k, type l of level $\Gamma_0(\mathfrak{p}\mathfrak{m})$, under some assumptions on the weight filtration of certain modular form associated to f. For $\mathfrak{m} = (1)$, we appealed to the structure theorem for Drinfeld modular forms of level $GL_2(A)$ to prove [DK1, Theorem 1.3]. For general $\mathfrak{m}$, a similar structure theorem for level $\Gamma_0(\mathfrak{m})$ is not available to prove [DK1, Theorem 1.5].

As an application of the main results of this article, we adopt the same methodology of $GL_2(A)$ for $\Gamma_0(T)$ to prove [DK1, Theorem 1.5] for level $\Gamma_0(\mathfrak{p}T)$. Recall that $\mathfrak{p}$ denotes a prime ideal of A generated by a monic irreducible polynomial

π of degree d such that $\mathfrak{p} \neq (T)$. Before stating the main result of this section we introduce some important operators.

Definition 6.1 (Θ-operator)**.** For Drinfeld modular forms, there is an analogue of Ramanujan's Θ-operator, which is defined as

$$\Theta := \frac{1}{\tilde{\pi}} \frac{d}{dz} = -u^2 \frac{d}{du}.$$

The Θ-operator preserves quasi-modularity but not modularity. However, one can perturb the Θ-operator to define the ∂_k-operator which preserves modularity.

Definition 6.2. For $k \in \mathbb{N}$ and $l \in \mathbb{Z}/(q-1)\mathbb{Z}$, we define $\partial_k : M_{k,l}(\Gamma_0(\mathfrak{n})) \to M_{k+2,l+1}(\Gamma_0(\mathfrak{n}))$ by

$$\partial_k f := \Theta f + kEf. \tag{6.1}$$

We shall write ∂ for ∂_k if the weight k is clear from the context.

Definition 6.3. The (partial) Atkin-Lehner operator is defined by the action of the matrix $W_{\mathfrak{p}}^{(\mathfrak{p}T)} := \left(\begin{smallmatrix} \pi & b \\ \pi T & d\pi \end{smallmatrix} \right)$ with $b, d \in A$ such that $d\pi^2 - b\pi T = \pi$. For more details, we refer the reader to [DK1, §3.1].

We now state the main result of this section.

Theorem 6.4. *Suppose* $f \in M_{k,l}^1(\Gamma_0(\mathfrak{p}T))$, $g \in M_{k+2,l+1}^1(\Gamma_0(\mathfrak{p}T))$ *have* $\mathfrak{p}$-*integral* u-*expansion at* ∞ *and satisfy* $\Theta f \equiv g \pmod{\mathfrak{p}}$. *Further, suppose that* $f|W_{\mathfrak{p}}^{(\mathfrak{p}T)} = \alpha f$, $g|W_{\mathfrak{p}}^{(\mathfrak{p}T)} = \beta g$ *with* $\alpha, \beta \in \{\pm 1\}$ *and* $(k, p) = 1$. *If* $w(\overline{F}) = (k-1)(q^d-1)+k$, *where* $F \in M_{(k-1)(q^d-1)+k,l}^1(\Gamma_0(T))$ *as in* [DK1, Proposition 3.8] *corresponding to* kf, *then* $\beta = -\alpha$.

For $\mathfrak{m} = (T)$ in [DK1, Theorem 1.5], it may appear that Theorem 6.4 is a special case, but there is a difference in definitions of weight filtration $w(\overline{F})$ in the respective theorems. We urge the reader to look at Definition 5.1 and [DK1, Definition 5.5] to note the difference. There the weight filtration was defined using the geometry of Drinfeld modular curves. Since the level of F is $\Gamma_0(T)$, we expect that these two definitions are same. However, we are unable to prove this.

The following proposition is a generalization of [Vin10, Theorem 3.1(1)] and is quite useful in the proof of Theorem 6.4.

Proposition 6.5. *Let* $\psi_d(U, V) \in A[U, V]$ *be the unique isobaric polynomial attached to* $c_T(\partial(g_d))$. *Then* $\overline{\psi}_d(U, V)$, $\overline{\phi}_d(U, V)$ *share no common factor.*

For $f \in M_{k,l}(\Gamma_0(T))_{A_\mathfrak{p}}$, let $\varphi_f(U, V) \in A_\mathfrak{p}[U, V]$ be the unique polynomial attached to $c_T(f)$ such that $f = \varphi_f(\Delta_W, \Delta_T)E_T^l$ (cf. Proposition 4.10). Define a derivation ∂ on $A_\mathfrak{p}[U, V, Z]$ (resp., on $\mathbb{F}_\mathfrak{p}[U, V, Z]$), by setting $\partial(U) := -UZ, \partial(V) := 0, \partial(Z) := Z^2$. By Proposition 4.3(7), we get $\vartheta(A_\mathfrak{p})(\partial(\varphi_f(U, V)Z^l)) = \partial_k(f)$, where ϑ is the mapping defined in Theorem 4.4.

Proof of Proposition 6.5. First, we note that the polynomials $\partial(\phi_d(U, V))$, $\psi_d(U, V)Z$ are equal. This follows by Proposition 4.10 and $\partial(\phi_d(\Delta_W, \Delta_T)) = \psi_d(\Delta_W, \Delta_T)E_T$.

If possible, let there be a common factor between $\overline{\psi}_d(U, V), \overline{\phi}_d(U, V)$, say a. Since $\overline{\phi}_d(U, V)$ is square-free, $\overline{\phi}_d(U, V) = ab$ for some b with $(a, b) = 1$. By Remark 4.14, the variables U and V do not divide $\overline{\phi}_d(U, V)$. Hence, we write

$$a = U^k + \sum_{i=1}^{k-1} \overline{c}_i U^{k-i} V^i + \overline{c}_k V^k \tag{6.2}$$

with $\overline{c}_k \neq 0$. This implies $\partial(a) = \eta(U, V)Z$, where

$$\eta(U, V) = (-kU^k + \sum_{i=1}^{k-1} -(k - i)\overline{c}_i U^{k-i} V^i).$$

Since $\overline{\psi}_d(U, V)Z = \partial(a)b + a\partial(b)$, $a \mid \overline{\psi}_d(U, V)$ implies $a \mid \partial(a)$ and hence $a \mid \eta(U, V)$. Considering the highest power of V in a and $\eta(U, V)$, we see that $a \mid \eta(U, V)$ can happen only if $\eta(U, V) = 0$. This forces that $k \equiv 0 \pmod{p}$ and $(k - i)\overline{c}_i = 0$ for all $1 \leq i \leq k - 1$. Hence, either $\overline{c}_i = 0$ or $i \equiv 0 \pmod{p}$ for all $1 \leq i \leq k - 1$. By (6.2), the polynomial a becomes a p-th power. This is a contradiction to $\overline{\phi}_d(U, V)$ being square-free. This proves the proposition. $\square$

The proof of Theorem 6.4 is similar to the proof of [DK1, Theorem 1.5]. Here, we use the structure theorem for $M^0(\Gamma_0(T))_R$ (i. e., Theorem 4.7).

6.1. **Proof of Theorem 6.4:** We shall prove this theorem by contradiction. Suppose $\beta = \alpha$. Let $E_{\mathfrak{p}}(z) = E(z) - \pi E(\pi z) \in M_{2,1}(\Gamma_0(\mathfrak{p}))$. We have the congruence $E_{\mathfrak{p}} \equiv E \pmod{\mathfrak{p}}$. Since $\Theta f \equiv g \pmod{\mathfrak{p}}$, (6.1) gives

$$\partial f \equiv g + kE_{\mathfrak{p}} f \pmod{\mathfrak{p}}.$$

Hence, there exists $h_1 \in M^1_{k+2,l+1}(\Gamma_0(\mathfrak{p}T))$ with $v_{\mathfrak{p}}(h_1) \geq 0$ such that

$$g - \partial f + kE_{\mathfrak{p}} f = \pi h_1. \tag{6.3}$$

Applying $W_{\mathfrak{p}}^{(\mathfrak{p}T)}$ on both sides, we obtain $\alpha(g - \partial f) = \pi h_1|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{p}T)}$ (cf. [DK1, Proposition 3.4] for more details). Combining this with (6.3), we get

$$kE_{\mathfrak{p}} f = \pi h_1 - \alpha \pi h_1|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{p}T)}$$

and hence $kE_{\mathfrak{p}} f \equiv -\alpha \pi h_1|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{p}T)} \pmod{\mathfrak{p}}$. By [DK1, Proposition 3.8], there exists $F \in M^1_{(k-1)q^d+1,l}(\Gamma_0(T))$ with $\mathfrak{p}$-integral u-expansion at ∞ such that $kf \equiv F \pmod{\mathfrak{p}}$. Hence

$$E_{\mathfrak{p}} F \equiv -\alpha \pi h_1|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{p}T)} \pmod{\mathfrak{p}} \tag{6.4}$$

By [DK1, Proposition 3.9], there exists $H \in M^1_{(k-1)q^d+3,l+1}(\Gamma_0(T))$ such that

$$H \equiv \alpha \pi h_1|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{p}T)} \pmod{\mathfrak{p}}.$$

Therefore (6.4) becomes $H \equiv -E_{\mathfrak{p}} F \pmod{\mathfrak{p}}$. Since $E_{\mathfrak{p}} \equiv E \pmod{\mathfrak{p}}$ and $E \equiv \partial(g_d) \pmod{\mathfrak{p}}$(cf. [Vin10, Theorem 1.1]), we get

$$H \equiv \partial(g_d) F \pmod{\mathfrak{p}}. \tag{6.5}$$

The last congruence implies that H has a $\mathfrak{p}$-integral u-expansion at ∞. Since both sides of (6.5) are congruent mod-$\mathfrak{p}$, we get

$$w(\overline{H}) = w(\overline{\partial(g_d)F}). \tag{6.6}$$

We now calculate the weight filtration for $\partial(g_d)F$. Note the weights of F, $\partial(g_d)F$ are $(k-1)(q^d-1)+k$, kq^d+2 respectively.

Let $\varphi_F(U,V)$ (resp., $\psi_d(U,V)$) be the unique isobaric polynomial attached to $c_T(F)$ (resp., to $c_T(\partial(g_d))$) such that $F = \varphi_F(\Delta_W,\Delta_T)E_T^l$ (resp., $\partial(g_d) = \psi_d(\Delta_W,\Delta_T)E_T$). Hence, $\partial(g_d)F = \varphi_F(\Delta_W,\Delta_T)\psi_d(\Delta_W,\Delta_T)E_T^{l+1}$ and

$$c_T(\partial(g_d)F) = \begin{cases} \frac{\partial(g_d)F}{E_T^{l+1}} & \text{if } 0 \le l < q-2, \\ \frac{\partial(g_d)F}{E_T^{l+1}}\Delta_W\Delta_T & \text{if } l = q-2. \end{cases}$$

Let $\eta(U,V)$ be the unique isobaric polynomial attached to $c_T(\partial(g_d)F)$. Then

$$\eta(U,V) = \begin{cases} \varphi_F(U,V)\psi_d(U,V) & \text{if } 0 \le l < q-2, \\ \varphi_F(U,V)\psi_d(U,V)UV & \text{if } l = q-2. \end{cases}$$

By Theorem 5.2(3), the assumption $w(\overline{F}) = (k-1)(q^d-1)+k$ implies $\overline{\phi}_d(U,V) \nmid \overline{\varphi}_F(U,V)$. By Proposition 6.5 and Remark 4.14, we get $\overline{\phi}_d(U,V) \nmid \overline{\eta}(U,V)$. This is true because the polynomials $\overline{\phi}_d(U,V)$, $\overline{\psi}_d(U,V)$ share no common factor and neither U nor V divide $\overline{\phi}_d(U,V)$. By Theorem 5.2(3), we get $w(\overline{\partial(g_d)F}) = kq^d+2$. Since the weight of H is $(k-1)q^d+3 < kq^d+2$, we conclude that

$$w(\overline{H}) \le (k-1)q^d+3 < kq^d+2 = w(\overline{\partial(g_d)F}),$$

which contradicts (6.6). This completes the proof.

Finally, we note that the condition on weight filtration $w(\overline{F})$ in Theorem 6.4 is automatically satisfied for doubly cuspidal forms of weight 2, type 1.

Corollary 6.6. *Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \neq (T)$. Suppose $f \in M_{2,1}^2(\Gamma_0(\mathfrak{p}T))$ and $g \in M_{4,2}^2(\Gamma_0(\mathfrak{p}T))$ have $\mathfrak{p}$-integral u-expansion at ∞ and satisfy $\Theta f \equiv g \pmod{\mathfrak{p}}$. Further, suppose that $f|W_{\mathfrak{p}}^{(\mathfrak{p}T)} = \alpha f$, $g|W_{\mathfrak{p}}^{(\mathfrak{p}T)} = \beta g$ with $\alpha, \beta \in \{\pm 1\}$. If $f \not\equiv 0 \pmod{\mathfrak{p}}$, then $\beta = -\alpha$.*

Proof. Arguing as in the proof of Theorem 6.4 for $f \in M_{2,1}^2(\Gamma_0(\mathfrak{p}T))$, we get $F \in M_{q^d+1,1}^2(\Gamma_0(T))$. Since $f \not\equiv 0 \pmod{\mathfrak{p}}$, the congruence $w(\overline{F}) \equiv q^d+1 \pmod{q^d-1}$(cf. Theorem 5.2(2)) implies that $w(\overline{F}) = 2$ or q^d+1. If $w(\overline{F}) = q^d+1$, then the corollary follows from Theorem 6.4. We now show that $w(\overline{F})$ cannot be 2.

Suppose $w(\overline{F}) = 2$. Since $M_{2,1}(\Gamma_0(T)) = \langle E_T \rangle$, we obtain $F \equiv cE_T \pmod{\mathfrak{p}}$ for some c. Comparing the coefficient of u on both sides of the congruence, we get $c \equiv 0 \pmod{\mathfrak{p}}$. Thus $F \equiv 0 \pmod{\mathfrak{p}}$, which is a contradiction. $\square$

Acknowledgments

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