

*Dedicated to Prof. Edriss S. Titi on the occasion of his 60th birthday*

**The role of BKM-type theorems in 3D Euler, Navier-Stokes and  
Cahn-Hilliard-Navier-Stokes analysis**

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**Abstract**

The Beale-Kato-Majda theorem contains a single criterion that controls the behaviour of solutions of the 3D incompressible Euler equations. Versions of this theorem are discussed in terms of the regularity issues surrounding the 3D incompressible Euler and Navier-Stokes equations together with a phase-field model for the statistical mechanics of binary mixtures called the 3D Cahn-Hilliard-Navier-Stokes (CHNS) equations. A theorem of BKM-type is established for the CHNS equations for the full parameter range. Moreover, for this latter set, it is shown that there exists a Reynolds number and a bound on the energy-dissipation rate that, remarkably, reproduces the  $Re^{3/4}$  upper bound on the inverse Kolmogorov length normally associated with the Navier-Stokes equations alone. An alternative length-scale is introduced and discussed, together with a set of pseudo-spectral computations on a  $128^3$  grid.

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## 1. Introduction

### 1.1. The 3D Euler, Navier-Stokes and Cahn-Hilliard-Navier-Stokes equations

The fine-scale turbulent dynamics, commonly observed in numerical simulations and experiments, has long been thought to be related to the issues concerning the regularity of solutions of both the 3D incompressible Euler and Navier-Stokes equations, although these issues remain largely unresolved [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Respectively, these equations are

$$(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p, \quad (1)$$

and

$$(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p + \mathbf{f}(\mathbf{x}). \quad (2)$$

In (1) and (2)  $\mathbf{u}$  is a divergence-free ( $\operatorname{div} \mathbf{u} = 0$ ) velocity field,  $\nu$  is the viscosity and  $\mathbf{f}(\mathbf{x})$  is a divergence-free, mean-zero,  $L^\infty$ -bounded forcing. In this paper the domain  $\mathcal{V}$  is taken to be a periodic box of side  $L$  and the uniform density  $\rho$  is set to unity.

Another system in which turbulent dynamics occurs is the phase-field model governed by the 3D Cahn-Hilliard equations. These are fundamental in the study of the statistical mechanics of binary mixtures [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]

$$\partial_t \phi = \gamma \Delta \mu, \quad (3)$$

where the chemical potential  $\mu = \delta \mathcal{F} / \delta \phi$  is related to the free energy

$$\mathcal{F} = \int_{\mathcal{V}} \left[ \frac{\Lambda}{2} |\nabla \phi|^2 + \frac{\Lambda}{4\xi^2} (\phi^2 - 1)^2 \right] dV. \quad (4)$$

$\mu$  is thus given by

$$\mu = \Lambda [-\Delta \phi + \xi^{-2} (\phi^3 - \phi)]. \quad (5)$$

This model can be used to study the mixing of two fluids, which are *immiscible* below a critical temperature, via a phase field  $\phi$ . In equilibrium,  $\phi = -1$  for one phase and  $\phi = 1$  for the other. The advantage of such a model is the continuity of the thin interface, of thickness  $\xi$ , between the two fluids. The existence of this interface removes the necessity of dealing with the complications of tracking a free boundary. When (3) is coupled to the 3D Navier-Stokes equations ( $\operatorname{div} \mathbf{u} = 0$ )

$$(\partial_t + \mathbf{u} \cdot \nabla) \phi = \gamma \Delta \mu, \quad \operatorname{div} \mathbf{u} = 0 \quad (6)$$

$$(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = \nu \Delta \mathbf{u} - \phi \nabla \mu - \nabla p + \mathbf{f}(\mathbf{x}), \quad (7)$$

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the combination of (5), (6) and (7) are known as the Cahn-Hilliard-Navier-Stokes (CHNS) equations. The parameter  $\gamma$  in (3) is called the mobility (Bray [17]), and  $\xi$  is the interface thickness. The interfacial dynamics are of especial interest, particularly regarding the immiscible Rayleigh-Taylor instability (RTI), which is manifest in this thin mixing layer: for references on the ubiquity of the RTI see [34, 35, 36, 37, 38, 39, 40, 41, 42]. Whether tightly-packed interfacial level sets remain continuous as time evolves is a question that is closely connected to the issue of the regularity of solutions, which remains an open problem for all these three sets of equations in three dimensions (3D). Various results are known in two dimensions (2D), such as the regularity of not only the 2D Navier-Stokes equations [6, 7, 10, 11, 12] but also of the stand-alone 2D Cahn-Hilliard equations (Elliott and Songmu [43]). The regularity problem for the 2D Cahn-Hilliard-Navier-Stokes (CHNS) equations has been solved in some remarkable papers by Abels [44, 45] and Gal and Grasselli [46] using different boundary conditions. In 3D, however, the issue remains a formidable open problem. Nevertheless, in the light of criteria that control their regularity, they do possess certain features in common with both the Euler and Navier-Stokes equations, and it is these that are the subject of this paper.

### 1.2. Statement of a theorem of BKM-type for the CHNS equations

The fundamental theorem that governs the behaviour of solutions of the 3D Euler equations is called the Beale-Kato-Majda (BKM) theorem [47]: see also Bardos and Titi [1] and Gibbon [3]. The statement of the theorem is simple. For  $n \geq 0$ , let us define

$$H_n = \int_{\mathcal{V}} |\nabla^n \mathbf{u}|^2 dV. \quad (8)$$

Now consider the vorticity  $\boldsymbol{\omega} = \text{curl } \mathbf{u}$ . The notation  $\|\cdot\|_p = (\int_{\mathcal{V}} |\cdot|^p dV)^{1/p}$  means that  $\|\boldsymbol{\omega}\|_\infty$  is the maximum or sup-norm of the vorticity in the domain  $\mathcal{V}$ .

**Theorem 1.** (Beale, Kato and Majda [47]) *For initial data of the 3D Euler equations satisfying  $u_0 \in H_n$  for  $n \geq 3$ , suppose there exists a solution on the interval  $[0, T^*)$  that loses regularity at the earliest time  $T^*$ , then*

$$\int_0^{T^*} \|\boldsymbol{\omega}\|_\infty d\tau = \infty. \quad (9)$$

*Conversely, if, for every  $T > 0$ ,  $\int_0^T \|\boldsymbol{\omega}\|_\infty d\tau < \infty$ , then solutions of the 3D Euler equations remain regular on  $[0, T]$ .*

The proof in [47] is short and the strategy is by contradiction. After some work, BKM found a differential inequality for  $H_n$ , in terms of  $\|\boldsymbol{\omega}\|_\infty$  which, when integrated in time up to and including  $T^*$ , proves that  $H_n$  is controlled from above

by  $\int_0^{T^*} \|\boldsymbol{\omega}\|_\infty d\tau$ . Given that the theorem presupposes that  $H_n$  loses regularity at  $T^*$ , we cannot have  $H_n(T^*) = \infty$  while  $\int_0^{T^*} \|\boldsymbol{\omega}\|_\infty d\tau$  remains finite.

Compared to the 3D Navier-Stokes equations, little is known about the behaviour<sup>1</sup> of solutions of the 3D Euler equations [1]. The value of the BKM theorem is that it furnishes us with a single, numerically testable criterion based on the behaviour of the time integral  $\int_0^T \|\boldsymbol{\omega}\|_\infty d\tau$ . There is a long history of numerical experiments that have aimed to test whether a singularity develops (see the list in [3]) but the latest work suggests that solutions do not blow up but undergo double exponential growth [51, 52]. The theorem also rules out potential algebraic singularities of a certain type: for instance, if one performs a numerical simulation and observes a singularity of the type  $\|\boldsymbol{\omega}\|_\infty \sim (T^* - t)^{-p}$ , then  $\int_0^{T^*} \|\boldsymbol{\omega}\|_\infty d\tau$  is finite for  $0 < p < 1$ . The theorem says that no singularity can occur, whereas the claim is that one has been observed. The ensuing contradiction can only be resolved by realizing that the observed singularity is an artefact of the numerical scheme employed. True singularities of this type must have  $p \geq 1$ .

Theorem 1 is specific to the 3D Euler equations and centres around the  $\|\boldsymbol{\omega}\|_\infty$ -criterion in (9), but it is possible to widen this idea to other model problems which display similar criteria for loss of regularity. These we will label as being of ‘‘BKM-type’’. In fact, two theorems of BKM-type that already been proved. The first is for the stochastic Euler equations by Crisan, Flandoli and Holm [53]. The second is a theorem similar to Theorem 1 that has already been proved by the authors in [54] for the 3D-CHNS equations, but with unit parameters only. One of the aims of this paper, among others, is to extend this proof to the full parameter range and to discuss its relationship with the versions valid for the 3D Euler and Navier-Stokes equations. Before stating it here, some background is necessary. The energy of the full CHNS system is given by (see Celani *et al.* [29])

$$E(t) = \int_{\mathcal{V}} \left\{ \frac{1}{2} \Lambda |\nabla \phi|^2 + \frac{\Lambda}{4\xi^2} (\phi^2 - 1)^2 + \frac{1}{2} |\mathbf{u}|^2 \right\} dV. \quad (10)$$

This is comprised of a sum of  $L^2$ -norms and clearly suggests an  $L^\infty$ -equivalent denoted as  $E_\infty$  and defined by

$$E_\infty(t) = \frac{1}{2} \Lambda \|\nabla \phi\|_\infty^2 + \frac{\Lambda}{4\xi^2} (\|\phi\|_\infty^2 - 1)^2 + \frac{1}{2} \|\mathbf{u}\|_\infty^2. \quad (11)$$

We also need a similar definition similar to  $H_n$  involving  $\phi$

$$P_n = \int_{\mathcal{V}} |\nabla^n \phi|^2 dV. \quad (12)$$

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<sup>1</sup>The wild solutions of De Lellis and Szekelyhidi [48, 49] lie in a category of their own: see also Buckmaster and Nicol [50].

The statement of the theorem for the full parameter range follows here below and its proof is discussed in §3.4 and Appendix B:

**Theorem 2.** *Consider the 3D CHNS equations on a periodic domain  $\mathcal{V} = [0, L]^3$ . For initial data  $u_0 \in H_n$ , for  $n \geq 2$ , and  $\phi_0 \in P_n$ , for  $n \geq 3$ , suppose there exists a solution on the interval  $[0, T^*)$ , where  $T^*$  is the earliest time that the solution loses regularity, then*

$$\int_0^{T^*} E_\infty(\tau) d\tau = \infty. \quad (13)$$

*Conversely, there exists a global solution of the 3D CHNS equation if, for every  $T > 0$ ,*

$$\int_0^T E_\infty(\tau) d\tau < \infty. \quad (14)$$

Clearly, this theorem is of BKM-type where  $E_\infty$  replaces  $\|\boldsymbol{\omega}\|_\infty$  in Theorem 1. As in the BKM theorem above, it provides us with a precise, single criterion for numerically monitoring the blow-up of solutions. Some types of blow-up could potentially be extremely subtle, such as a cusp forming in a tightly packed level sets in the CHNS-interface; this could potentially cause a high derivative to become singular. These are ruled out if  $\int_0^t E_\infty d\tau < \infty$ . However, an obvious question to ask is why the Navier-Stokes part of  $E_\infty$  is proportional to  $\|\mathbf{u}\|_\infty^2$  and not  $\|\boldsymbol{\omega}\|_\infty$ ? This question is answered in §2 where several well-known 3D Navier-Stokes regularity criteria are summarized (see Table 1) and where it is shown that while  $E_\infty$ -theorem is akin to the Euler equations in being of BKM-type, the  $\|\mathbf{u}\|_\infty^2$  term has its origins in the Navier-Stokes equations. This is followed by a section on the CHNS equations, in which some new results on bounds for the energy dissipation rate in terms of the Reynolds number are displayed.

In the original proof of the  $E_\infty$ -theorem in [54], the parameters  $\nu$ ,  $\Lambda$ ,  $\gamma$  and  $\xi$  were set to unity for convenience. In §3, dimensional analysis is used to create a new version of the proof with the full parameter range.

Finally, thanks to our state-of-the-art direct numerical simulations (DNSs) in §3.2, we have been able to monitor the complete time series of the energy dissipation rate and thus calculate the mean dissipation rate. These DNSs also help us to estimate a new alternative length scale based on  $\sqrt{\Lambda}$ . This helps us to see if there is (or is not) any ordering of the conventional length scale and the new alternative scale; this cannot be predicted analytically.

## 2. Regularity properties of the 3D Navier-Stokes equations: the $\int_0^t \|\mathbf{u}\|_\infty^2 d\tau$ criterion

The structure of  $E_\infty$  in Theorem 2 is intriguing and raises the question why the Navier-Stokes contribution is of the form  $\|\mathbf{u}\|_\infty^2$  and not the conventional

$\|\boldsymbol{\omega}\|_\infty$ . The first subsection discusses this question while the second summarizes current knowledge of the boundedness of time-averages, particularly the energy dissipation rate which is of relevance when this issue is raised for the 3D CHNS equations in §3.1.

2.1.  $\int_0^t \|\mathbf{u}\|_\infty^2 d\tau$  as a Navier-Stokes regularity criterion

3D Navier-Stokes and Euler regularity are substantially different in that pointwise control in time over  $H_1$  is sufficient for the existence and uniqueness of solutions of the 3D Navier-Stokes equations whereas this is insufficient for the 3D Euler equations which require the finiteness of  $\|\boldsymbol{\omega}\|_\infty$ . To look further at this, let us formally differentiate<sup>2</sup>  $H_1$  with respect to time to obtain :

$$\frac{1}{2}\dot{H}_1 \leq -\nu H_2 + \left| \int_{\mathcal{V}} \boldsymbol{\omega} \cdot (\boldsymbol{\omega} \cdot \nabla \mathbf{u}) dV \right| + \|\mathbf{f}\|_2 H_1^{1/2}. \quad (15)$$

There are two ways of estimating the central integral term :

$$\left| \int_{\mathcal{V}} \boldsymbol{\omega} \cdot (\boldsymbol{\omega} \cdot \nabla \mathbf{u}) dV \right| \leq \begin{cases} \|\boldsymbol{\omega}\|_\infty H_1, \\ \|\mathbf{u}\|_\infty H_1^{1/2} H_2^{1/2}. \end{cases} \quad (16)$$

With the first estimate, (15) becomes

$$\frac{1}{2}\dot{H}_1 \leq -\nu H_2 + \|\boldsymbol{\omega}\|_\infty H_1 + \|\mathbf{f}\|_2 H_1^{1/2}, \quad (17)$$

and with the second,

$$\frac{1}{2}\dot{H}_1 \leq -\frac{1}{2}\nu H_2 + \frac{1}{2}\nu^{-1} \|\mathbf{u}\|_\infty^2 H_1 + \|\mathbf{f}\|_2 H_1^{1/2}. \quad (18)$$

Dropping the negative  $H_2$ -terms in both (17) and (18) it is clear that  $H_1(t)$  is bounded from above provided either

$$\int_0^t \|\boldsymbol{\omega}\|_\infty d\tau < \infty \quad \text{or} \quad \int_0^t \|\mathbf{u}\|_\infty^2 d\tau < \infty. \quad (19)$$

The first is obviously the BKM criterion of Theorem 1, valid for both the 3D Euler and Navier-Stokes equations, but the second is valid only for 3D Navier-Stokes because of the role played by the viscous term in deriving (18). It is the second criterion that appears naturally in  $E_\infty$ , as the proof in Appendix B shows.

The alternative criterion,  $\int_0^t \|\mathbf{u}\|_\infty^2 d\tau < \infty$ , displayed in (19) has a place in the broader class of regularity criteria due to Serrin (see [8])

$$u \in L^p(0, T; L^q), \quad 2/p + 3/q = 1. \quad (20)$$

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<sup>2</sup>See [7, 8] for a more rigorous weak solution approach.

What is known	What is sufficient for regularity
$\ \mathbf{u}(\cdot, t)\ _2 < \infty$	$\ \mathbf{u}(\cdot, t)\ _3 < \infty$
$\int_0^t \ \mathbf{u}\ _\infty d\tau < \infty$	$\int_0^t \ \mathbf{u}\ _\infty^2 d\tau < \infty$
$\int_0^t H_1 d\tau < \infty$	$\int_0^t H_1^2 d\tau < \infty$
$\int_0^t \ \boldsymbol{\omega}\ _\infty^{1/2} d\tau < \infty$	$\int_0^t \ \boldsymbol{\omega}\ _\infty d\tau < \infty$

Table 1: Table of the results that are known (left column) for the 3D Navier-Stokes equations and those results that are sufficient for regularity but unproved (right column). The notation is  $\|\cdot\|_p = (\int_{\mathcal{V}} |\cdot|^p dV)^{1/p}$ . The results  $\int_0^t \|\mathbf{u}\|_\infty d\tau < \infty$  and  $\int_0^t \|\boldsymbol{\omega}\|_\infty^{1/2} d\tau < \infty$  are both due to Guillopé, Foias, and Temam [57].

Table 1 displays a set of Navier-Stokes regularity criteria, the first row of which contains the  $\|\mathbf{u}(\cdot, t)\|_3 < \infty$  criterion of Escauriaza, Seregin and Sverak [9]: it is clear that this criterion lies at one end of (20) with  $p = \infty$  and  $q = 3$ , while the  $\int_0^t \|\mathbf{u}\|_\infty^2 d\tau < \infty$  criterion, which lies in the second row, lies at the other where  $p = 2$  and  $q = \infty$ . Finally we note that there are other Navier-Stokes regularity criteria that lie outside this class: for instance, those based on the pressure [56].

## 2.2. Bounded time averages

The Navier-Stokes equations possess a well known energy inequality that goes back to Leray [5]. It takes the form<sup>3</sup>

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{V}} |\mathbf{u}|^2 dV \leq -\nu H_1 + \|\mathbf{u}\|_2 \|\mathbf{f}\|_2. \quad (21)$$

The  $\mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u})$  term vanishes under the Divergence Theorem. With a time average defined by

$$\langle \cdot \rangle_T = \frac{1}{T} \int_0^T \cdot d\tau \quad (22)$$

and with an average velocity  $U$  and a box frequency  $\varpi_0$  defined by

$$U^2 = L^{-3} \langle \|\mathbf{u}\|_2^2 \rangle_T, \quad \varpi_0 = \nu L^{-2}, \quad (23)$$

and with Grashof and Reynolds numbers defined as

$$Gr = \frac{L^{3/2} \|\mathbf{f}\|_2}{\nu^2}, \quad Re = \frac{UL}{\nu}, \quad (24)$$

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<sup>3</sup>The mathematical statements in this section are purely formal: for a full weak solution exposition based on Leray's weak solutions [5], see [6, 7, 8].

a time average of (21) gives

$$\langle H_1 \rangle_T \leq \varpi_0^2 L^3 Gr Re + O(T^{-1}) . \quad (25)$$

Doering and Foias [55] have shown that, for a forcing function with a single scale  $\ell$ , for which we take  $\ell = L$  for convenience, then  $Gr \leq c Re^2$ , where the dimensionless constant  $c$  is a function of the shape of the forcing. Thus (25) becomes

$$\langle H_1 \rangle_T \leq c \varpi_0^2 L^3 Re^3 + O(T^{-1}) , \quad (26)$$

and the energy-dissipation rate  $\mathcal{E}$  is bounded by

$$\mathcal{E} = \nu L^{-3} \langle H_1 \rangle_T , \quad (27)$$

and so we end up with the classic estimate for the inverse Kolmogorov length  $\lambda_k^{-1}$

$$L \lambda_k^{-1} = (\mathcal{E}/\nu^3)^{1/4} \quad \Rightarrow \quad L \lambda_k^{-1} \leq c Re^{3/4} . \quad (28)$$

This type of  $Re^{3/4}$ -estimate is reflected in similar results for the 3D CHNS equations displayed below.

### 3. The 3D CHNS equations

#### 3.1. New estimates on the energy dissipation rate

Consider the energy of the full CHNS equations as in Celani, *et al.* [29] stated earlier in (10)

$$E(t) = \int_{\mathcal{V}} \left\{ \frac{1}{2} \Lambda |\nabla \phi|^2 + \frac{\Lambda}{4\xi^2} (\phi^2 - 1)^2 + \frac{1}{2} |\mathbf{u}|^2 \right\} dV . \quad (29)$$

Then a formal differentiation gives

$$\begin{aligned} \frac{dE}{dt} &= \int_{\mathcal{V}} \{ \Lambda \nabla \phi \cdot \nabla [-\mathbf{u} \cdot \nabla \phi + \gamma \Delta \mu] + (\mu + \Lambda \Delta \phi) (-\mathbf{u} \cdot \nabla \phi + \gamma \Delta \mu) \} dV \\ &+ \int_{\mathcal{V}} \{ \mathbf{u} \cdot (\nu \Delta \mathbf{u} - \phi \nabla \mu - \nabla p + \mathbf{f}) \} dV . \end{aligned} \quad (30)$$

Integrating by parts on the first term and using the property  $\text{div } \mathbf{u} = 0$  and the Divergence Theorem to remove the pressure term, we find

$$\begin{aligned} \frac{dE}{dt} &= -\Lambda \int_{\mathcal{V}} \Delta \phi [-\mathbf{u} \cdot \nabla \phi + \gamma \Delta \mu] dV \\ &+ \int_{\mathcal{V}} (\mu + \Lambda \Delta \phi) (-\mathbf{u} \cdot \nabla \phi + \gamma \Delta \mu) dV \\ &- \int_{\mathcal{V}} \phi \mathbf{u} \cdot \nabla \mu dV + \int_{\mathcal{V}} (-\nu |\nabla \mathbf{u}|^2 + \mathbf{u} \cdot \mathbf{f}) dV . \end{aligned} \quad (31)$$



Parameter	Dimension
$[\nu]$	$L^2T^{-1}$
$[\xi]$	$L$
$[\Lambda]$	$L^4T^{-2}$
$[\gamma]$	$T$
$[\phi]$	none
$[\mu]$	$L^2T^{-2}$

Table 2: Dimensions of the various parameters and the variable  $\mu$  in (3) – (5).

Terms in the first and second line of (31) cancel. Moreover,  $\int_{\mathcal{V}} \mathbf{u} \cdot \nabla(\phi\mu) dV = 0$  because  $\text{div } \mathbf{u} = 0$ , leaving us with

$$\frac{dE}{dt} = - \int_{\mathcal{V}} (\nu |\nabla \mathbf{u}|^2 + \gamma |\nabla \mu|^2) dV + \int_{\mathcal{V}} \mathbf{u} \cdot \mathbf{f} dV. \quad (32)$$

Thus, without any additive forcing,  $dE/dT < 0$ , in which case  $E$  decays, a result which is true in every dimension. For the forced case we can do better with time averages  $\langle \cdot \rangle_T$  up to time  $T$  defined in (22). Define the full energy-dissipation rate as

$$\mathcal{E}(\nu, \gamma) = L^{-3} \left\langle \int_{\mathcal{V}} (\nu |\nabla \mathbf{u}|^2 + \gamma |\nabla \mu|^2) dV \right\rangle_T, \quad (33)$$

then a time average of (32) gives

$$\mathcal{E}(\nu, \gamma) \leq U \|\mathbf{f}\|_2 + O(T^{-1}) = \nu^3 L^{-4} Gr Re + O(T^{-1}), \quad (34)$$

where the average velocity  $U$  is defined in (23). Defining a Kolmogorov-like length in the conventional way

$$L\lambda_{\nu, \gamma}^{-1} = (\mathcal{E}(\nu, \gamma)/\nu^3)^{1/4}, \quad (35)$$

we can appeal to the modified Doering-Foias relation between  $Gr$  and  $Re$  proved in the Appendix and shown earlier in (27). Remarkably, this still stands for the CHNS system with minor modifications. Thus we have

$$L\lambda_{\nu, \gamma}^{-1} \leq c Re^{3/4}. \quad (36)$$

### 3.2. An alternative length scale

Because  $\sqrt{\Lambda}$  and  $\nu$  have the same dimensions it is also possible to define new variables  $\mathcal{E}_\Lambda$ ,  $L\lambda_{\Lambda, \gamma}^{-1}$  in which  $\nu$  has been replaced by  $\sqrt{\Lambda}$ . Thus we define

$$\mathcal{E}(\Lambda, \gamma) = L^{-3} \left\langle \int_{\mathcal{V}} \left( \sqrt{\Lambda} |\nabla \mathbf{u}|^2 + \gamma |\nabla \mu|^2 \right) dV \right\rangle_T, \quad (37)$$

with

$$L\lambda_{\Lambda,\gamma}^{-1} = \left( \mathcal{E}(\Lambda, \gamma) / \Lambda^{3/2} \right)^{1/4}. \quad (38)$$

Which is the smallest scale,  $\lambda_{\nu,\gamma}$  or  $\lambda_{\Lambda,\gamma}$ ? The answer depends on how large, in relative terms, are the integrals in (33) and (37), and how large, in relative terms, are  $\nu$  and  $\sqrt{\Lambda}$ ?

To investigate these two scales, we turn to direct numerical simulations (DNSs) of the 3D CHNS equations, using constant energy injection and a constant forcing wave number. We use a pseudospectral method, with  $N/2$  dealiasing because of the cubic nonlinearity and, for the purpose of illustration,  $128^3$  collocation points. We use a second-order Adams-Bashforth method for time marching. In our CHNS description, both components of the fluids have the same viscosity; and we assume that  $\gamma$  is independent of  $\phi$ . Other details of such a DNS can be found in Refs.[30, 54] which we follow here.

In the way  $\lambda_{\nu,\gamma}$  and  $\lambda_{\Lambda,\gamma}$  have been defined, their relative value will depend only on  $\sqrt{\Lambda}$  and  $\nu$ . R1-R8 in Table 3, remarkably show that, for the parameters we have used,

$$\lambda_{\nu,\gamma} \leq \lambda_{\Lambda,\gamma}. \quad (39)$$

Only if  $\nu = \gamma = \sqrt{\Lambda}$ , do we find  $\lambda_{\nu,\gamma} = \lambda_{\Lambda,\gamma}$  (see Run R7-R8). In Figs. 1-3, we have shown the time series of  $\gamma |\nabla\mu|^2$  (left panel),  $|\phi|^2$  (middle panel), and  $\nu|\nabla u|^2$  (right panel) for run R1, R3 and R7 respectively. For runs R3, R4, R7, R8 we have chosen a small value of the mixing-energy density, which leads to high mixing and is responsible for the vanishingly small values of  $\gamma |\nabla\mu|^2$  shown in the insets of the left panels of Figs. 2 and 3.

	$\langle Re_\lambda \rangle_t$	$\nu$	$\Lambda$	$\gamma$	$\nu \langle \ \nabla\mathbf{u}\ _2^2 \rangle_t$	$\gamma \langle \ \nabla\mu\ _2^2 \rangle_t$	$\lambda_{\nu,\gamma}$	$\lambda_{\Lambda,\gamma}$
R1	27	$1.16 \times 10^{-2}$	0.107703	$1.16 \times 10^{-2}$	0.2418	0.1078	1.1462	6.6594
R2	27	$1.16 \times 10^{-2}$	0.107703	$6.25 \times 10^{-3}$	0.2440	0.1063	1.1456	6.6447
R3	30	$1.16 \times 10^{-2}$	0.0016	$1.16 \times 10^{-2}$	0.3496	0	1.1462	2.1285
R4	30	$1.16 \times 10^{-2}$	0.0016	$6.25 \times 10^{-3}$	0.3493	0	1.1458	2.1286
R5	48	$5.00 \times 10^{-3}$	0.0707	$5.00 \times 10^{-3}$	0.1910	0.1589	0.6096	5.1519
R6	47	$5.00 \times 10^{-3}$	0.107703	$5.00 \times 10^{-3}$	0.2184	0.1317	0.6095	5.5441
R7	21	$2.00 \times 10^{-2}$	$4.00 \times 10^{-4}$	$2.00 \times 10^{-2}$	0.3500	0	1.7249	1.7249
R8	13	$4.00 \times 10^{-2}$	$1.60 \times 10^{-3}$	$4.00 \times 10^{-2}$	0.3505	0	2.8987	2.8987

Table 3: The entries in the table are values of the parameters  $Re_\lambda$ ,  $\nu$ ,  $\Lambda$ ,  $\gamma$ ,  $\nu \langle \|\nabla\mathbf{u}\|_2^2 \rangle_t$ ,  $\gamma \langle \|\nabla\mu\|_2^2 \rangle_t$ ,  $\lambda_{\nu,\gamma}$ , and  $\lambda_{\Lambda,\gamma}$  for our DNS runs R1-R8. The number of collocation points is kept fixed at  $N = 128$  in each direction (so the total number of collocation points is  $128^3$ ). The forcing wave numbers are fixed at  $k_f = 1\&2$ ,  $\nu$  is the kinematic viscosity, the Cahn number  $Ch = \xi/L$ , where  $\xi$  is the interface width, is kept fixed at  $Ch = 0.01$  (with  $L = 2\pi$ ), and  $Re_\lambda$  is the Taylor-microscale Reynolds number. In all cases  $\langle \cdot \rangle_t$  denotes the average over time in the statistically steady state.

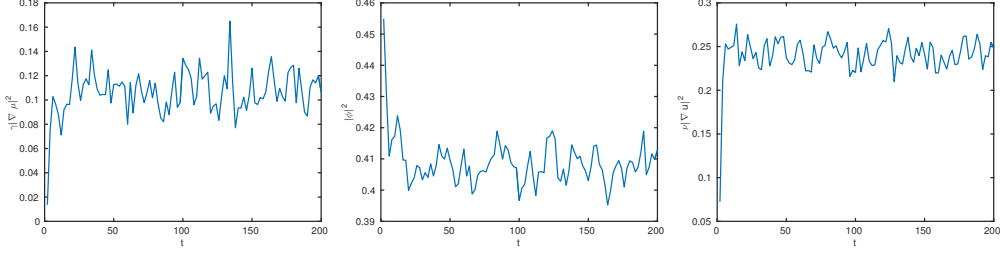


Figure 1: Plots versus time  $t$  of  $\gamma \|\nabla \mu\|_2^2$  (left panel),  $\|\phi\|_2^2$  (middle panel), and  $\nu \|\nabla \mathbf{u}\|_2^2$  (right panel) for run R7. The labelling in the figures is  $|\cdot|^2 \equiv \|\cdot\|_2^2$ .

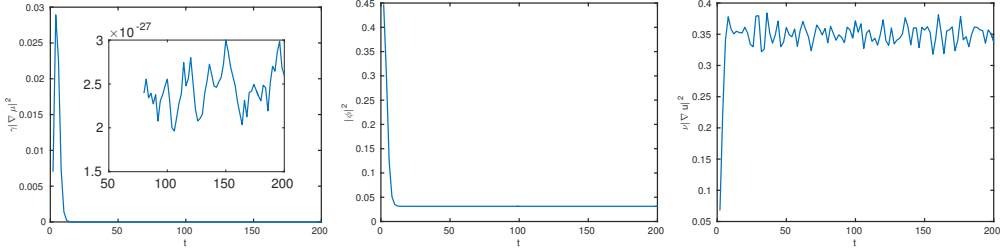


Figure 2: Plots versus time  $t$  of  $\gamma \|\nabla \mu\|_2^2$  (left panel),  $\|\phi\|_2^2$  (middle panel), and  $\nu \|\nabla \mathbf{u}\|_2^2$  (right panel) for run R3. The inset in the left panel shows the plot of  $\gamma \|\nabla \mu\|_2^2$  versus time  $t$ ; this shows that  $\gamma \|\nabla \mu\|_2^2$  is vanishingly small for this run, in the statistically steady state. As in Fig. 1, the labelling in the figures is  $|\cdot|^2 \equiv \|\cdot\|_2^2$ .

### 3.3. Dimensionless forms of the CHNS equations

Let us transform the CHNS equations into dimensionless (primed) coordinates beginning with a characteristic velocity  $U$  and using the layer thickness  $\xi$  as the characteristic length (see Table 2)

$$t' = U\xi^{-1}t, \quad x' = \xi^{-1}x, \quad \mathbf{u}' = U^{-1}\mathbf{u}, \quad \phi' = \phi. \quad (40)$$

Moreover, let

$$\mu' = -\Delta' \phi' + \phi'^3 - \phi', \quad \text{with} \quad \mu = \Lambda \xi^{-2} \mu', \quad (41)$$

and let the dimensionless pressure  $p'$  be defined as  $p' = (\tau \xi^{-1})^2 p$ . Then, in dimensionless form, the CHNS equations become (with  $\text{div}' \mathbf{u}' = 0$ )

$$(\partial_{t'} + \mathbf{u}' \cdot \nabla') \phi' = S_1^{-1} \Delta' \mu', \quad (42)$$

$$(\partial_{t'} + \mathbf{u}' \cdot \nabla') \mathbf{u}' = S_2^{-1} \Delta' \mathbf{u}' - S_3^{-1} \phi' \nabla' \mu' - \nabla' p' + \mathbf{f}', \quad (43)$$

where  $S_1^{-1}$ ,  $S_2^{-1}$  and  $S_3^{-1}$  are dimensionless parameters:

$$S_1^{-1} = \frac{\Lambda \gamma}{U \xi^3}, \quad S_2^{-1} = \frac{\nu}{U \xi}, \quad S_3^{-1} = \frac{\Lambda}{U^2 \xi^2}. \quad (44)$$

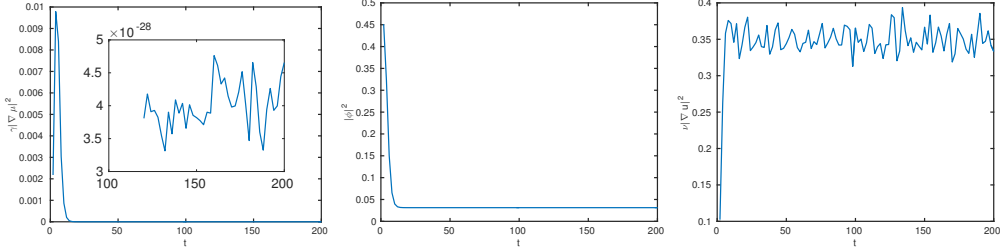


Figure 3: Plots versus time  $t$  of  $\gamma\|\nabla\mu\|_2^2$  (left panel),  $\|\phi\|_2^2$  (middle panel), and  $\nu\|\nabla\mathbf{u}\|_2^2$  (right panel) for run R7. The inset in the left panel shows the plot of  $\gamma\|\nabla\mu\|_2^2$  versus time  $t$ ; this shows that  $\gamma\|\nabla\mu\|_2^2$  is vanishingly small for this run, in the statistically steady state. The labelling in the figures is  $|\cdot|^2 \equiv \|\cdot\|_2^2$ .

The cubic domain is  $[0, Ch^{-1}]^3$  where  $Ch = \xi/L$  is the Cahn number which represents the interface thickness normalized with the characteristic length scale (here the characteristic length scale is the box-size  $L = 2\pi$ ). In the CHNS literature, the Péclet number  $Pe$  is also used, which is the ratio of the convective and diffusive time scales. It is also the product of the Reynolds number and the Schmidt number. For characteristic length and velocity scales  $L$  and  $U$ ,  $Pe = LU/D$ , where  $D = \gamma\Lambda/\xi^2$  is the diffusivity. If the CHNS equation is non-dimensionalized by using  $\xi$  as the characteristic length scale, then  $Pe = U\xi^3/\Lambda\gamma = S_1$ .

We can also write  $S_3 = CaRe_\xi$ , where  $Ca = \rho\nu U/\sigma$  is the capillary number and  $Re_\xi = U\xi/\nu$  is the Reynolds number at the length scale  $\xi$ ; and  $\sigma = 22\Lambda/3\xi \Rightarrow S_3 \sim \rho U^2 \xi^2/\Lambda$ .

### 3.4. Proof of Theorem 2 for the full set of parameters

The number of parameters  $S_i^{-1}$  ( $i = 1, 2, 3$ ) in (42) significantly lengthens and complicates the proof of Theorem 2 so in [54] this was performed for unit parameters: this is repeated in Appendix B for completeness. In this section we show that there is a way of adapting the unit-parameter proof to the full set of parameter values. Let us return to the energy

$$\begin{aligned}
 E(t) &= \int_{\mathcal{V}} \left\{ \frac{1}{2}\Lambda|\nabla\phi|^2 + \frac{\Lambda}{4\xi^2}(\phi^2 - 1)^2 + \frac{1}{2}|\mathbf{u}|^2 \right\} dV \\
 &= \xi\Lambda \int_{\mathcal{V}} \left\{ \frac{1}{2}|\nabla'\phi'|^2 + \frac{1}{4}(\phi'^2 - 1)^2 + \frac{1}{2}S_3|\mathbf{u}'|^2 \right\} dV' \\
 &\equiv \xi\Lambda E'(t),
 \end{aligned} \tag{45}$$

where, based on the definition of  $E'(t)$  in (45), we define its  $L^\infty$ -equivalent

$$E'_\infty = \frac{1}{2}\|\nabla'\phi'\|_\infty^2 + \frac{1}{4}(\|\phi'\|_\infty^2 - 1)^2 + \frac{1}{2}S_3\|\mathbf{u}'\|_\infty^2. \tag{46}$$

and thus  $E'_\infty = \Lambda^{-1}\xi^2 E_\infty$  with  $E_\infty$  defined in (11). Indeed, the calculation leading to (32) can be repeated using the dimensionless form  $E'$  above. In the following we develop a strategy based upon  $E'_\infty$  defined in (46). In [54] a BKM-type theorem was proved with the various parameters set to unity for convenience, which makes  $S_1 = S_2 = S_3$ . This proof in [54] can be used to prove the theorem for the full parameter set by using a device. Firstly, it is easy to prove that

$$\min \{1, S_3^{-1}\} E'_\infty \leq E'_{unit} \leq \max \{1, S_3^{-1}\} E'_\infty. \quad (47)$$

where, with unit variables,

$$E'_{unit} = \frac{1}{2}\|\nabla'\phi'\|_\infty^2 + \frac{1}{4}(\|\phi'\|_\infty^2 - 1)^2 + \frac{1}{2}\|\mathbf{u}'\|_\infty^2. \quad (48)$$

The direction of the inequalities in the proof allow us to prove the theorem in the primed variables with unit parameter values (i.e.  $E'_{unit}$ ), and then, using (47), replace  $E'_{unit}$  with  $E'_\infty$ , which translates back to  $E_\infty$  in dimensional variables. Thus we have proved the theorem for all positive values of the parameters in  $E_\infty$ .

#### 4. Conclusion

The regularity problem for the 3D CHNS equations is a hard problem: it compounds the formidable difficulties found when addressing the same issue in its two constituent parts, namely the 3D Navier-Stokes and 3D CHNS equations respectively. While there are also clear parallels with the Navier-Stokes problem, which suggests a Leray-type approach to weak solutions might be fruitful, various difficulties stand in the way. For instance, in (32) the usual bound on the time average of  $\nu \int_{\mathcal{V}} |\nabla \mathbf{u}|^2 dV$  in the energy dissipation rate is the root of all the results for the Navier-Stokes part, but we also have the Cahn-Hilliard contribution of  $\gamma \int_{\mathcal{V}} |\nabla \mu|^2 dV$  on the right hand side. Finding estimates in terms of this is a difficult task and one that currently lies out of reach.

For the present we have to be content with a Beale-Kato-Majda-type of result as displayed in Theorem 2. We could now call this a class of theorems as there are now three of its type: (i) the original for the 3D Euler equations; (ii) our Theorem 2 for the CHNS equations, and (iii) a theorem for the stochastic 3D Euler equations of Crisan, Flandoli and Holm [53]. The  $128^3$  simulations display no evidence of singular behaviour although, computationally, this is a very demanding problem and requires further investigation.

The structure of the energy dissipation in (32) has allowed us to introduce an  $Re^3$  energy bound and thus a  $Re^{3/4}$  upper bound on  $(L\lambda_{\nu,\gamma})^{-1}$ , this inverse length scale being defined in (35). One further interesting result arising in the simulations is the fact that for both  $\nu > \sqrt{\Lambda}$  and  $\nu < \sqrt{\Lambda}$  we see an ordering in these two length scales such that  $\lambda_{\nu,\gamma} < \lambda_{\Lambda,\gamma}$ , a result for which we see no evidence analytically. This, again, requires further investigation.

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## Appendix A. The Doering-Foias relation between $Gr$ and $Re$

Doering and Foias [55] split the forcing function  $\mathbf{f}(\mathbf{x})$  into its magnitude  $F$  and its “shape”  $\phi$  such that

$$\mathbf{f}(\mathbf{x}) = F\phi(\ell^{-1}\mathbf{x}), \quad (\text{A.1})$$

where  $\ell$  is the longest length scale in the force but here it is taken to be  $\ell = L$  for convenience. On the unit torus  $\mathbb{I}_d$ ,  $\phi$  is a mean-zero, divergence-free vector field with the chosen normalization property

$$\int_{\mathbb{I}_d} |\nabla_y^{-1}\phi|^2 d^d y = 1. \quad (\text{A.2})$$

$L^2$ -norms of  $\mathbf{f}$  on  $\mathbb{I}^d$  are

$$\|\nabla^N \mathbf{f}\|_2^2 = C_N \ell^{-2N} L^d F^2, \quad (\text{A.3})$$

where the coefficients  $C_N$  refer to the shape of the force but not its magnitude

$$C_M = \sum_n |2\pi n|^{2N} |\hat{\phi}_n|^2. \quad (\text{A.4})$$

Doering and Foias [55] showed that various bounds exist such as (among others)

$$\|\nabla \Delta^{-M} \mathbf{f}\|_\infty = D_M F \ell^{2M-1}. \quad (\text{A.5})$$

The energy-dissipation rate  $\epsilon$  is

$$\epsilon = \left\langle \nu L^{-d} \int_{\mathbb{I}_d} |\nabla \mathbf{u}|^2 dV \right\rangle = \nu L^{-d} \langle H_1 \rangle. \quad (\text{A.6})$$

In terms of  $F$  the Grashof number in (24) becomes

$$Gr = F \ell^3 / \nu^2 \quad (\text{A.7})$$

and the Taylor micro-scale  $\lambda_T$  is related to  $U$  via  $\lambda_T = \sqrt{\nu U^2 / \epsilon}$ , which is consistent with the definition  $\lambda_T^{-2} = \langle H_1 \rangle / \langle H_0 \rangle$ .

Following the procedure in [10] (page 296 equation (2.9)) we multiply the Navier-Stokes equations by  $(-\Delta^{-M})\mathbf{f}$  to obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{I}_d} \mathbf{u} \cdot [(-\Delta^{-M})\mathbf{f}] dV &= \nu \int_{\mathbb{I}_d} \Delta \mathbf{u} \cdot [(-\Delta^{-M})\mathbf{f}] \\ &- \int_{\mathbb{I}_d} \nabla^{-M} \mathbf{f} \cdot \nabla^{-M} \mathbf{f} dV \\ &- \int_{\mathbb{I}_d} \mathbf{u} \cdot \nabla \mathbf{u} \cdot [(-\Delta^{-M})\mathbf{f}] dV \\ &+ \int_{\mathcal{V}} \phi (\Delta^{-M} \mathbf{f}) \cdot \nabla \mu dV, \end{aligned} \quad (\text{A.8})$$

where the pressure term vanishes in the usual way. Now integrate all the terms by parts, and take the time average

$$\begin{aligned} \left\langle \int_{\mathbb{I}_d} \nabla^{-M} \mathbf{f} \cdot \nabla^{-M} \mathbf{f} dV \right\rangle_T &\leq \nu \left\langle \int_{\mathbb{I}_d} \mathbf{u} \cdot [(-\Delta^{-M+1})\mathbf{f}] dV \right\rangle_T \\ - \left\langle \int_{\mathbb{I}_d} \mathbf{u} \cdot [\nabla [(-\Delta^{-M})\mathbf{f}]] \cdot \mathbf{u} dV \right\rangle_T &+ \left\langle \int_{\mathbb{I}_d} \phi \Delta^{-M} \mathbf{f} \cdot \nabla \mu \right\rangle_T. \end{aligned} \quad (\text{A.9})$$

An extra term  $\int_{\mathcal{V}} \phi \Delta^{-M} \mathbf{f} \cdot \nabla \mu dV$  derives from the  $-\phi \nabla \mu$ -term in (3). However, all its contributions are zero except one, given the definition of  $\mu$ . (A.9) becomes

$$c_0 F^2 \ell^{2M} \leq c_1 \nu F \ell^{2M-2} U + c_2 \ell^{2M-1} F U^2 + c_3 \Lambda^{-1} \ell^{2M-1} F \langle E \rangle_T, \quad (\text{A.10})$$

where the  $U^2$ -term contains the contributions from both nonlinear terms and the constants (not explicitly given) contain the shape of the body forcing. Using (A.7), as  $Gr \rightarrow \infty$ , (A.10) becomes

$$Gr \leq c_4 (Re + Re^2) + O(\langle E \rangle_T). \quad (\text{A.11})$$

## Appendix B. Proof of Theorem 2 with unit parameters

In the following the parameters in the dimensionless system  $S_n$  are set to unity and primes have been removed<sup>4</sup>. The domain is now the cube  $[0, Ch^{-1}]^3$ . Then, we recall the definitions of  $H_n$  in equation (8) and  $P_n$  in equation (12) and proceed in 3 steps.

**Step 1:** We begin with the time evolution of  $P_n$  (the dot above  $P_n$  denotes a time derivative),:

$$\frac{1}{2} \dot{P}_n = -P_{n+2} + P_{n+1} + \int_{\mathcal{V}} (\nabla^n \phi) \nabla^n \Delta (\phi^3) dV - \int_{\mathcal{V}} (\nabla^n \phi) \nabla^n (\mathbf{u} \cdot \nabla \phi) dV; \quad (\text{B.1})$$

<sup>4</sup>This proof has been published in [54] but is included here for completeness.

and then we estimate the third term on the right as

$$\left| \int_{\mathcal{V}} (\nabla^n \phi) \nabla^n \Delta(\phi^3) dV \right| \leq \|\nabla^n \phi\|_2 \sum_{i,j=0}^{n+2} C_{i,j}^{n+2} \|\nabla^i \phi\|_p \|\nabla^j \phi\|_q \|\nabla^{n+2-i-j} \phi\|_r, \quad (\text{B.2})$$

where  $1/p + 1/q + 1/r = 1/2$ . Now we use a sequence of Gagliardo-Nirenberg inequalities

$$\begin{aligned} \|\nabla^i \phi\|_p &\leq c_{n,i} \|\nabla^{n+2} \phi\|_2^{a_1} \|\phi\|_\infty^{1-a_1}, \\ \|\nabla^j \phi\|_q &\leq c_{n,j} \|\nabla^{n+2} \phi\|_2^{a_2} \|\phi\|_\infty^{1-a_2}, \\ \|\nabla^n \phi\|_r &\leq c_{n,i,j} \|\nabla^{n+2-i-j} \phi\|_2^{a_3} \|\phi\|_\infty^{1-a_3}, \end{aligned} \quad (\text{B.3})$$

where, in  $d$  dimensions,

$$\begin{aligned} \frac{1}{p} &= \frac{i}{d} + a_1 \left( \frac{1}{2} - \frac{n+2}{d} \right), \\ \frac{1}{q} &= \frac{j}{d} + a_2 \left( \frac{1}{2} - \frac{n+2}{d} \right), \\ \frac{1}{r} &= \frac{n+2-i-j}{d} + a_3 \left( \frac{1}{2} - \frac{n+2}{d} \right). \end{aligned} \quad (\text{B.4})$$

By summing these and using  $1/p + 1/q + 1/r = 1/2$ , it is seen that  $a_1 + a_2 + a_3 = 1$ . Thus, we have

$$\left| \int_{\mathcal{V}} (\nabla^n \phi) \nabla^{n+2}(\phi^3) dV \right| \leq c_n \|\nabla^n \phi\|_2 \|\nabla^{n+2} \phi\|_2 \|\phi\|_\infty^2 \leq \frac{1}{2} P_{n+2} + c_n P_n \|\phi\|_\infty^4, \quad (\text{B.5})$$

and so Eq. (B.1) becomes (here and henceforth coefficients such as  $c_n$  are multiplicative constants),

$$\frac{1}{2} \dot{P}_n = -\frac{1}{2} P_{n+2} + P_{n+1} + c_n \|\phi\|_\infty^4 P_n + \left| \int_{\mathcal{V}} (\nabla^n \phi) \nabla^n (\mathbf{u} \cdot \nabla \phi) dV \right|. \quad (\text{B.6})$$

Estimating the last term in Eq. (B.6) we have

$$\begin{aligned} \left| \int_{\mathcal{V}} (\nabla^n \phi) \nabla^n (\mathbf{u} \cdot \nabla \phi) dV \right| &= \left| - \int_{\mathcal{V}} (\nabla^{n+1} \phi) \nabla^{n-1} (\mathbf{u} \cdot \nabla \phi) dV \right| \\ &\leq \|\nabla^{n+1} \phi\|_2 \sum_{i=0}^{n-1} C_i^n \|\nabla^i \mathbf{u}\|_p \|\nabla^{n-1-i} (\nabla \phi)\|_q, \end{aligned} \quad (\text{B.7})$$

where  $1/p + 1/q = 1/2$ . Now we use two Gagliardo-Nirenberg inequalities in  $d$  dimensions to obtain

$$\|\nabla^i \mathbf{u}\|_p \leq c \|\nabla^{n-1} \mathbf{u}\|_2^a \|\mathbf{u}\|_\infty^{1-a}, \quad (\text{B.8})$$

$$\|\nabla^{n-1-i} (\nabla \phi)\|_q \leq c \|\nabla^{n-1} (\nabla \phi)\|_2^b \|\nabla \phi\|_\infty^{1-b}. \quad (\text{B.9})$$



Equations (B.8) and (B.9) follow from

$$\frac{1}{p} = \frac{i}{d} + a \left( \frac{1}{2} - \frac{n-1}{d} \right), \quad (\text{B.10})$$

$$\frac{1}{q} = \frac{n-1-i}{d} + b \left( \frac{1}{2} - \frac{n-1}{d} \right). \quad (\text{B.11})$$

Because  $1/p + 1/q = 1/2$  then  $a + b = 1$ . Thus the second term in Eq. (B.1) turns into<sup>5</sup>

$$\begin{aligned} \left| \int_{\mathcal{V}} (\nabla^n \phi) \nabla^n (\mathbf{u} \cdot \nabla \phi) dV \right| &\leq c_n P_{n+1}^{1/2} H_{n-1}^{a/2} P_n^{b/2} \|\mathbf{u}\|_{\infty}^{1-a} \|\nabla \phi\|_{\infty}^{1-b} \quad (\text{B.12}) \\ &\leq P_{n+1}^{1/2} [c_n H_{n-1} \|\nabla \phi\|_{\infty}^2]^{a/2} [P_n \|\mathbf{u}\|_{\infty}^2]^{b/2} \\ &\leq \frac{1}{2} P_{n+1} + \frac{1}{2} a c_n H_{n-1} \|\nabla \phi\|_{\infty}^2 + \frac{1}{2} b P_n \|\mathbf{u}\|_{\infty}^2, \end{aligned}$$

and Eq. (B.6) becomes

$$\frac{1}{2} \dot{P}_n = -\frac{1}{2} P_{n+2} + \frac{3}{2} P_{n+1} + c_{n,1} \left( \frac{1}{2} \|\phi\|_{\infty}^4 + \|\mathbf{u}\|_{\infty}^2 \right) P_n + c_{n,2} H_{n-1} \|\nabla \phi\|_{\infty}^2. \quad (\text{B.13})$$

**Step 2:** Now we look at  $H_n$  defined in Eq. (8) using Eq. (2) with  $\mathbf{f} = -\hat{z}\phi$ . The easiest way is to use the 3D NS equation in the vorticity form as in Doering and Gibbon [10] where gradient terms have been absorbed into the pressure term, which disappears under the curl-operation

$$(\partial_t + \mathbf{u} \cdot \nabla) \boldsymbol{\omega} = \Delta \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nabla \phi \times \nabla \Delta \phi - \nabla^{\perp} \phi. \quad (\text{B.14})$$

Therefore, following the methods used in [10], we find

$$\begin{aligned} \frac{1}{2} \dot{H}_n &\leq -\frac{1}{2} H_{n+1} + c_n \|\mathbf{u}\|_{\infty}^2 H_n + \left| \int_{\mathcal{V}} (\nabla^{n-1} \boldsymbol{\omega}) [\nabla^{n-1} (\nabla \phi \times \Delta \nabla \phi)] dV \right| \\ &+ \left| \int_{\mathcal{V}} (\nabla^{n-1} \boldsymbol{\omega}) [\nabla^{n-1} \nabla^{\perp} \phi] dV \right|. \quad (\text{B.15}) \end{aligned}$$

Beginning with the third term on the right-hand side of Eq. (B.15), we obtain

$$\left| \int_{\mathcal{V}} (\nabla^{n-1} \boldsymbol{\omega}) \nabla^{n-1} (\nabla \phi \times \Delta \nabla \phi) dV \right| \leq \|\nabla^{n-1} \boldsymbol{\omega}\|_2 \sum_{i=0}^{n-1} C_i^n \|\nabla^i (\nabla \phi)\|_r \|\nabla^{n+1-i} (\nabla \phi)\|_s. \quad (\text{B.16})$$

Then, by using a Gagliardo-Nirenberg inequality,

$$\|\nabla^i (\nabla \phi)\|_r \leq c \|\nabla^{n+1} (\nabla \phi)\|_2^a \|\nabla \phi\|_{\infty}^{1-a}, \quad (\text{B.17})$$

$$\|\nabla^{n+1-i} (\nabla \phi)\|_s \leq c \|\nabla^{n+1} (\nabla \phi)\|_2^b \|\nabla \phi\|_{\infty}^{1-b}, \quad (\text{B.18})$$

---

<sup>5</sup>Inequalities (B.8) and (B.12) are the origin of the  $\|\mathbf{u}\|_{\infty}^2$ -term in  $E_{\infty}$ .

where  $1/r + 1/s = 1/2$  and where

$$\frac{1}{r} = \frac{i}{d} + a \left( \frac{1}{2} - \frac{n+1}{d} \right) \quad (\text{B.19})$$

$$\frac{1}{s} = \frac{n+1-i}{d} + b \left( \frac{1}{2} - \frac{n+1}{d} \right), \quad (\text{B.20})$$

we find that  $a + b = 1$ . This yields

$$\begin{aligned} \left| \int_{\mathcal{V}} (\nabla^{n-1} \boldsymbol{\omega}) \nabla^{n-1} (\nabla \phi \times \Delta \nabla \phi) dV \right| &\leq c_n H_n^{1/2} P_{n+2}^{1/2} \|\nabla \phi\|_{\infty} \\ &\leq P_{n+2} + \frac{1}{4} c_n H_n \|\nabla \phi\|_{\infty}^2. \end{aligned} \quad (\text{B.21})$$

The last term on the right-hand side of Eq. (B.15) is easily handled. Altogether we find

$$\frac{1}{2} \dot{H}_n \leq -\frac{1}{2} H_{n+1} + P_{n+2} + c_{n,3} (\|\mathbf{u}\|_{\infty}^2 + \|\nabla \phi\|_{\infty}^2) H_n + \frac{1}{2} H_n + \frac{1}{2} P_n \quad (\text{B.22})$$

**Step 3:** Finally, by noting that  $X_n = P_{n+1} + H_n$ , we use Eq. (B.5) with  $n \rightarrow n+1$  to obtain

$$\begin{aligned} \frac{1}{2} \dot{X}_n &\leq -\frac{1}{2} P_{n+3} + \frac{3}{2} P_{n+2} + c_{n,1} \left( \frac{1}{2} \|\phi\|_{\infty}^4 + \|\mathbf{u}\|_{\infty}^2 \right) P_{n+1} + c_{n,2} H_n \|\nabla \phi\|_{\infty}^2 \\ &\quad - \frac{1}{2} H_{n+1} + P_{n+2} + c_{n,3} (\|\mathbf{u}\|_{\infty}^2 + \|\nabla \phi\|_{\infty}^2) H_n + \frac{1}{2} H_n + \frac{1}{2} P_n \\ &\leq -\frac{1}{2} P_{n+3} - \frac{1}{2} H_{n+1} + \frac{5}{2} P_{n+2} + c_{n,4} \left( \frac{1}{2} \|\phi\|_{\infty}^4 + \|\mathbf{u}\|_{\infty}^2 + \|\nabla \phi\|_{\infty}^2 \right) X_n \\ &\quad + \frac{1}{2} H_n + \frac{1}{2} P_n. \end{aligned} \quad (\text{B.23})$$

By using  $P_{n+2} \leq P_{n+3}^{1/2} P_{n+1}^{1/2} \leq (\varepsilon/2) P_{n+3} + (1/2\varepsilon) P_{n+1}$ , with  $\varepsilon$  chosen as  $\varepsilon = \frac{1}{5}$ , we have (with  $P_n \leq Ch^{-2} P_{n+1}$ )

$$\frac{1}{2} \dot{X}_n \leq -\frac{1}{4} P_{n+3} - \frac{1}{2} H_{n+1} + c_{n,4} \max(1, Ch^{-2}) (\|\nabla \phi\|_{\infty}^2 + \frac{1}{2} \|\phi\|_{\infty}^4 + \|\mathbf{u}\|_{\infty}^2 + \frac{1}{2}) X_n. \quad (\text{B.24})$$

We note that  $\phi$  is a mean-zero function on our periodic domain  $[0, Ch^{-1}]^3$ , so  $\|\phi\|_{\infty} \leq Ch^{-1} \|\nabla \phi\|_{\infty}$ . Then we can write

$$\begin{aligned} c_{n,5} \left( \|\nabla \phi\|_{\infty}^2 + \frac{1}{2} \|\phi\|_{\infty}^4 + \|\mathbf{u}\|_{\infty}^2 + \frac{1}{2} \right) &= \\ c_{n,5} \left( \|\nabla \phi\|_{\infty}^2 + \frac{1}{2} (\|\phi\|_{\infty}^2 - 1)^2 + \|\mathbf{u}\|_{\infty}^2 + \|\phi\|_{\infty}^2 \right) & \\ \leq 2c_{n,5} \left( \|\nabla \phi\|_{\infty}^2 + \frac{1}{2} (\|\phi\|_{\infty}^2 - 1)^2 + \|\mathbf{u}\|_{\infty}^2 \right). & \end{aligned} \quad (\text{B.25})$$

By dropping the negative terms, Eq. (B.24) turns into

$$\frac{1}{4} \dot{X}_n \leq c_{n,5} E'_{unit} X_n, \quad (\text{B.26})$$

where  $E'_{unit}$  is defined in Eq. (47) which has unit parameters. This can then be replaced by  $E'_\infty$  using the same inequality. By integrating over  $[0, T^*]$ , we obtain

$$X_n(T^*) \leq c_{n,6} X_n(0) \exp \int_0^{T^*} E'_\infty(\tau) d\tau. \quad (\text{B.27})$$

Clearly, having  $\int_0^{T^*} E'_\infty < \infty$  contradicts the statement in the Theorem that solutions first lose regularity at  $T^*$ . Translating back to dimensional variables we have the result. ■

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