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Start-up shear flow of a shear-thinning fluid that approximates the response of viscoplastic fluids

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A B S T R A C T

In this paper we study a start-up shear flow of a recently proposed model of a shear-thinning fluid that mimics the response of a class of viscoplastic materials, namely the flow between parallel plates, one of which is fixed and the other is started impulsively. In simple shear flows while the generalized viscosity blows up, the shear stress is yet finite and thus the fluid is able to mimic the response of viscoplastic fluids. The analytical solution of the velocity profile is obtained using the semi-inverse approach for a perturbation approximation. The partial differential equation for the perturbation approximation is solved numerically, and compared with the analytical solution in order to validate the numerical scheme. The full equations are then solved numerically and the effect of the various material moduli are assessed by introducing appropriate dimensionless variables.

1. Introduction

Many materials, based on the time scale of observation, seem to resist flowing until a critical shear stress is reached, and such materials are referred to as Bingham fluids (a sub-class of viscoplastic materials) in virtue of the seminal work of Bingham [1] concerning such materials. Such fluid behavior is relevant to many practical situations and hence such constitutive relations have been studied in great detail. It would be impossible to refer to even a small sub-set of such studies and so we rest content mentioning a few relevant studies where the reader could find more references (see Bird et al. [2], Mitsoulis [3], de Souza Mendes and Thompson [4], Papanastasiou and Boudouvis [16], Fusi et al. [6], Farina and Fusi [7], Fusi et al. [8], Huilgol [9]). However, if by fluid one means a body that cannot resist shear, then a fluid that presents a yield stress would be a contradiction in terms. The question then that one needs to address is if, near zero shear rate, the viscosity of the fluid is so high that its flow is not observable if the time scale of observation is not sufficiently large, without there being a clear-cut yield stress. The answer is in the affirmative, and in this short paper we study the unsteady response of a fluid that seems to mimic the response of a viscoplastic fluid (see Garimella et al. [10]).

Papanastasiou[11] developed a fluid model to mimic the response of materials that exhibit a "yield stress". While the constitutive relation introduced by Papanastasiou [11] exhibits a great deal of similarity to our model, there are however fundamental differences, both with regard to the rationale espoused by Papanastasiou for introducing the constitutive relation, and with respect to the response displayed by the two constitutive relations. Papanastasiou's rationale for developing the model is heavily reliant on the material having a yield stress and he constantly refers to notions such as the "yield

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stress" and "yield surface", etc., in his discussion of the development of the constitutive relation. His aim is to obtain a constitutive relation to approximate what he perceives as a material with a "yield stress". Our aim is not to describe the response of viscoplastic materials, but to construct a constitutive relation that describes a fluid (a material that cannot resist shear, and does not have a yield stress) that seems to mimic the response of a viscoplastic material. As the title of the paper in which the constitutive relation was introduced states, we have developed "A new model to describe the response a class of seemingly viscoplastic materials". The critical term "seemingly" implies that the response of materials under consideration are not viscoplastic materials but seem to suggest that they are. The models are for materials that do not exhibit "yield stress". Next, the constitutive relation introduced by Papanastasiou has three constants, and more importantly when the shear rate tends to infinity the material behaves like a Navier-Stokes fluid, that is the generalized viscosity does not continue to decrease in value and has a limiting viscosity. The constitutive relation that we study has one less material parameter that is needed to characterize it, and its generalized viscosity continues to decrease with increasing shear rate and at very high shear rates it behaves like a Euler fluid (a fluid with no viscosity). The shear stress tends to infinity with the shear rate tending to infinity in the Papanastasiou model, while in our constitutive relation the shear stress cannot become unbounded, it has limited shear stress.

We study the unsteady flow between two parallel plates, one plate is suddenly started and the other is stationary, of a non-Newtonian fluid that has been shown to mimic reasonably well the response of a viscoplastic fluid (see Garimella et al. [10]). The above problem is a variant of problems studied by Stokes in the case of Navier-Stokes fluid (see Stokes [12]; Stokes considered the flow of a fluid atop an infinite plate due to a sudden accelerating plate). The above variant of Stokes' first problem has been considered by Rajagopal [13] for fluids of second grade, and by Srinivasan and Rajagopal [14] for a fluid with pressure dependent viscosity.

Our proposed (see Garimella et al. [10]) constitutive relation does not exhibit a threshold for the stress , that is, it is not a viscoplastic fluid. However, its generalized viscosity as a function of the symmetric part of the velocity gradient is such that it has a sufficiently steep gradient for the generalized viscosity in terms of the shear rate in that it mimics the response of viscoplastic fluids. The model is essentially that of a shear-thinning fluid with a limiting shear stress, and it has a sufficiently large viscosity at zero shear rate so as to approximate a material having "yield stress". We have determined the material moduli that characterize the constitutive relation by corroborating the same against experimental data, and predictions of flows based on our model agreed with the data on viscoplastic fluids.

The paper is organized as follows. The governing equations are described in Section 2. The problem formulation for the flow between two infinite parallel plates is described in Section 3. We seek an analytical solution for the equations by using a perturbation approximation and we obtain the same using the semi-inverse approach in Section 4. We compare the analytical solution that we establish against the numerical solution obtained using our numerical procedure in Section 5. Having ascertained the validity of the numerical procedure, we then solve the full equation numerically, and document the effect of model parameters on the features of the flow in Section 5. We summarize the results, and give recommendations for future work in Section 6.

2. Governing equations

The mass balance equation is:

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{v}) = 0, \tag{1}$$

where $\frac{\partial}{\partial t}$ is the partial derivative with respect to time, ρ is the fluid density, $div()$ denotes the divergence operator, and $\mathbf{v} = v_x \boldsymbol{e}_{\mathrm{x}} + v_y \boldsymbol{e}_{\mathrm{y}} + v_z \boldsymbol{e}_{\mathrm{z}}$ is the velocity vector with the components of the velocity v_x, v_y and v_z, in the directions x, y and z, respectively.

The linear momentum balance equation is:

$$\rho \frac{D\mathbf{v}}{Dt} = div\mathbf{T} + \rho \mathbf{b}, \tag{2}$$

where $\frac{D}{Dt}$ is the Lagrangian time derivative, which is given by $\frac{\partial}{\partial t}$ + grad(.)**v, T** is the Cauchy stress tensor, and **b** is the specific body force.

3. Problem formulation

We consider a shear thinning fluid lying between two infinitely long parallel plates separated by a distance h. The Cartesian coordinate system (x, y, z) is used, and the plate lies on the x-z plane, and the plate is moving along the x-direction, while the y- axis is normal to the plates. Initially, both the plates and fluid are at rest. The top plate is kept stationary, and the flow is generated by a sudden movement of the bottom plate along the x-axis with a constant velocity, v_0, applied at time $t = 0$. We assume that the flow is unsteady and unidirectional, and that the fluid is incompressible. The schematic representation of the flow domain with coordinates (x, y) is shown in Fig. 1.

We assume that the constitutive equation for the fluid is given by

$$\mathbf{T} = -p\mathbf{I} + 2\mu(\|\mathbf{D}\|)\mathbf{D}, \tag{3}$$

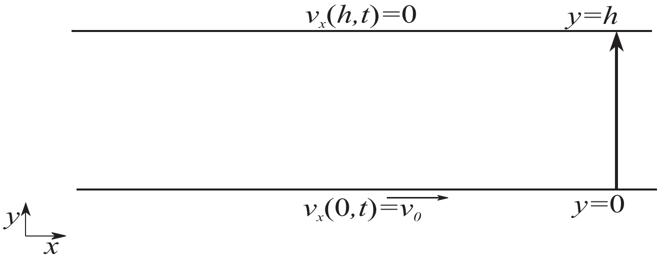

Fig. 1. Flow between a fixed plate and an impulsively started plate.

where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, p is the mechanical pressure, $\mathbf{v}$ denotes the velocity, $\mathbf{D}$ denotes the symmetric part of the velocity gradient defined through

$$\mathbf{D} = \frac{\left(\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{T}}\right)}{2}, \tag{4}$$

and $\mu(\|\mathbf{D}\|)$ is the generalized viscosity which is dependent on the symmetric part of the velocity gradient $\mathbf{D}$, where $\|\mathbf{D}\|$ is the Frobenius norm of the symmetric part of the velocity gradient.

We consider the following form for the generalised viscosity:

$$\mu(\|\mathbf{D}\|) = \frac{\alpha_1}{\|\mathbf{D}\|} - \frac{\alpha_1 \exp(-\alpha_2\|\mathbf{D}\|)}{\|\mathbf{D}\|}, \tag{5}$$

where $\alpha_1,\ \alpha_2$ are positive constants, and

$$\|\mathbf{D}\| = \sqrt{\mathrm{trace}\left(\mathbf{D}^2\right)}. \tag{6}$$

It is important to note that in simple shear flow, while the viscosity given by (Eq. (5)) blows up, the shear stress is finite. It is the fact that the viscosity blows up that allows the model to mimic a viscoplastic fluid.

4. Analytical solution for a perturbation approximation

In this section, we use the semi-inverse approach with a perturbation approximation to solve the problem analytically.

The assumed velocity field and pressure field are of the form

$$\mathbf{v} = v_x(y,t)\boldsymbol{e}_x, \ \ p = p(y,t). \tag{7}$$

The above velocity field automatically satisfies the mass balance equation, and the balance of linear momentum (by neglecting gravity) simplifies to

$$\rho\frac{\partial v_x}{\partial t} = \frac{\partial \tau_{yx}}{\partial y}, \tag{8}$$

$$-\frac{\partial p}{\partial y} = 0. \tag{9}$$

The initial condition for the velocity is

$$v_x(y,\ t) = 0 \quad \forall \ \ t \le 0. \tag{10}$$

The boundary conditions for the velocity are

$$v_x(0,\ t) = v_0, \tag{11}$$

$$v_x(h,\ t) = 0. \tag{12}$$

It would be appropriate at this juncture to point to a very interesting issue that concerns incompatibility between the initial condition expressed in (10) and the boundary condition (11). In the limit of $t = 0$, the conditions (10) and (11) are not compatible. This is to be expected as the plate is impulsively started at $t = 0$ and there is a discontinuity. This issue is discussed at length in Bandelli et al. [15] in their discussion of Stokes' second problem in the context of a different non-Newtonian fluid.

The symmetric part of the velocity gradient $\mathbf{D}$ simplifies, for the assumed flow field, to

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial v_x}{\partial y} & 0 \\ \frac{\partial v_x}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

Subsequently, the Cauchy stress tensor (Eq. (3)) is expressed as

$$\mathbf{T}=\begin{bmatrix} -p & \mu(\|\mathbf{D}\|)\frac{\partial v_x}{\partial y} & 0 \\ \mu(\|\mathbf{D}\|)\frac{\partial v_x}{\partial y} & -p & 0 \\ 0 & 0 & -p \end{bmatrix}. \tag{14}$$

Thus

$$\tau_{yx}=\mu(\|\mathbf{D}\|)\frac{\partial v_x}{\partial y}. \tag{15}$$

We introduce dimensionless variables given below, indicated with asterisks, to non-dimensionalise the governing equations and express the solution characteristics. These are:

$$y^*=\frac{y}{h}, v_x^*=\frac{v_x}{v_0}, t^*=\frac{tv_0}{h}, \mu^*(\|\mathbf{D}\|)=\frac{\mu(\|\mathbf{D}\|)}{\mu_0},$$

$$p^*=\frac{ph}{\mu_0 v_0}, \alpha_1^*=\frac{\alpha_1 h}{\mu_0 v_0}, \alpha_2^*=\frac{\alpha_2 v_0}{h}, Re=\frac{\rho v_0 h}{\mu_0},$$

where h is the distance between the parallel plates, ρ is the reference density, μ_0 is the reference viscosity, v_0 is the reference velocity, and Re is the Reynolds number.

Employing the dimensionless variables, we obtain the governing Eqs. ((8), (9)) in the following form

$$\frac{\partial v_x^*}{\partial t^*}=\frac{1}{Re}\frac{\partial \tau_{yx}^*}{\partial y^*}, \tag{16}$$

$$-\frac{\partial p^*}{\partial y^*}=0. \tag{17}$$

The initial and boundary conditions given in Eqs. ((10)–(12)) are given below in dimensionless form.

The initial condition for the velocity is

$$v_x^*(y^*,\ t^*)=0 \quad \forall \quad t^*\leq 0. \tag{18}$$

The boundary conditions for the velocity are

$$v_x^*(0,\ t^*)=1, \tag{19}$$

$$v_x^*(1,\ t^*)=0. \tag{20}$$

Eq. (15) for the shear stress is expressed in dimensionless form as

$$\tau_{yx}^*=\mu^*(\|\mathbf{D}\|)\frac{\partial v_x^*}{\partial y^*}. \tag{21}$$

Eqs. (4) and (6) yield

$$\|\mathbf{D}\|=\sqrt{\mathrm{trace}\left(\mathbf{D^2}\right)}=\frac{1}{\sqrt{2}}\left|\frac{\partial v_x}{\partial y}\right|=\frac{1}{\sqrt{2}}\frac{v_0}{h}\left|\frac{\partial v_x^*}{\partial y^*}\right|. \tag{22}$$

Substituting Eqs. (5), (22) and (21) in the governing Eq. (16), we obtain

$$\frac{\partial v_x^*}{\partial t^*}=\frac{1}{Re}\frac{\partial}{\partial y^*}\left(-\sqrt{2}\alpha_1^*+\sqrt{2}\alpha_1^*\exp\left(\frac{\alpha_2^*}{\sqrt{2}}\frac{\partial v_x^*}{\partial y^*}\right)\right), \tag{23}$$

leading to

$$\frac{\partial v_x^*}{\partial t^*}=\frac{\sqrt{2}\alpha_1^*}{Re}\frac{\partial}{\partial y^*}\left(\exp\left(\frac{\alpha_2^*}{\sqrt{2}}\frac{\partial v_x^*}{\partial y^*}\right)\right). \tag{24}$$

We will solve equation (24) subject to Eqs. (18)-(20) numerically, but in order to validate the numerical procedure, we shall establish an analytical solution to a perturbation approximation, and once it is seen to be efficacious we use it to solve the full system numerically.

Using the Taylor series expansion - $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots$ - and by retaining the first two terms, Eq. (24) simplifies to

$$\frac{\partial v_x^*}{\partial t^*}=\frac{\alpha_1^*\alpha_2^*}{Re}\frac{\partial^2 v_x^*}{\partial y^{*2}}. \tag{25}$$

We write the solution $v_x^*(y^*,\ t^*)$ in terms of a new variable $u^*(y^*,\ t^*)$, such that $u^*(y^*,\ t^*)$ has homogeneous boundary conditions.

The new variable $u^*(y^*,\ t^*)$ is assumed to be

$$u^*(y^*,\ t^*) = v^*_x(y^*,\ t^*) - v^*_{x,SS}(y^*), \tag{26}$$

where $v^*_{x,SS}(y^*)$ is the steady state solution for the velocity.

The final governing equation Eq. (25) reduces, for the steady state, to

$$\frac{d^2 v^*_{x,SS}}{dy^{*2}} = 0. \tag{27}$$

whose solution is

$$v^*_{x,SS} = c\ y^* + d, \tag{28}$$

where c and d are constants.

The boundary conditions for the velocity are

$$v^*_{x,SS}(0) = 1, \tag{29}$$

$$v^*_{x,SS}(1) = 0. \tag{30}$$

By applying these boundary conditions to Eq. (28), we obtain

$$v^*_{x,SS}(y^*) = 1 - \ y^*. \tag{31}$$

Substituting Eq. (31) in the new variable given by Eq. (26), we get

$$u^*(y^*,\ t^*) = v^*_x(y^*,\ t^*) - 1 + \ y^*. \tag{32}$$

Substituting Eq. (32) in Eq. (25), the final governing equation in the variable $u^*(y^*,\ t^*)$ is given as

$$\frac{\partial u^*}{\partial t^*} = \frac{\alpha_1^* \alpha_2^*}{Re} \frac{\partial^2 u^*}{\partial y^{*2}}. \tag{33}$$

The initial condition for $u^*(y^*,\ t^*)$ is

$$u^*(y^*,\ 0) = v^*_x(y^*, 0) - v^*_{x,SS}(y^*) = -1 + \ y^*. \tag{34}$$

The boundary conditions for $u^*(y^*,\ t^*)$ are

$$u^*(0,\ t^*) = 0, \tag{35}$$

$$u^*(1,\ t^*) = 0. \tag{36}$$

The boundary conditions are now homogeneous.

It follows that the solution is given by

$$u^*_n(y^*, t^*) = E_n \sin(n\ \pi\ y^*) e^{-n^2\pi^2\ A\ t^*}. \tag{37}$$

We now use the initial condition of $u^*(y^*,\ t^*)$ to find E_n. A standard procedure leads to:

$$E_n = -\frac{2}{n\pi}. \tag{38}$$

Substituting Eqs. (37) and (38) in Eq. (32), the final velocity profile $v^*_x(y^*,\ t^*)$ is expressed as

$$v^*_x(y^*,\ t^*) = (1 - y^*) + \sum_{n=1}^{\infty} \left(-\frac{2}{n\pi}\right) \sin(n\ \pi\ y^*)\ e^{-n^2\pi^2\ A\ t^*}. \tag{39}$$

This approximate analytical solution will be used to test the numerical procedure which will be used to solve the partial differential equations.

5. Numerical solution

First, we numerically solve the equation governing the approximate Eq. (25) subject to the initial condition Eq. (18) and boundary conditions Eqs. ((19) and (20)) and compare it against the exact analytical solution to the same approximate equation to determine the accuracy and the efficacy of the numerical method. Having ascertained the efficacy of the numerical procedure, we use it to solve the full partial differential equation Eq. (24) subject to the appropriate initial and boundary conditions numerically. We use the pdepe solver available in MATLAB R2020a software.

The model parameters, α_1 and $\alpha_2$are obtained by fitting the equation for the shear stress predicted by the model with the experimental data [2,3] using curve fitting tool (cftool) in MATLAB as detailed previously in Garimella et al. [10]. The constants obtained are (for kaolin-water: α_1=1.44 $\times 10^2$ N/m^2, α_2=1.58 $\times 10^{-3}$ s), (for meat extract: α_1=1.54 $\times 10^2$ N/m^2,

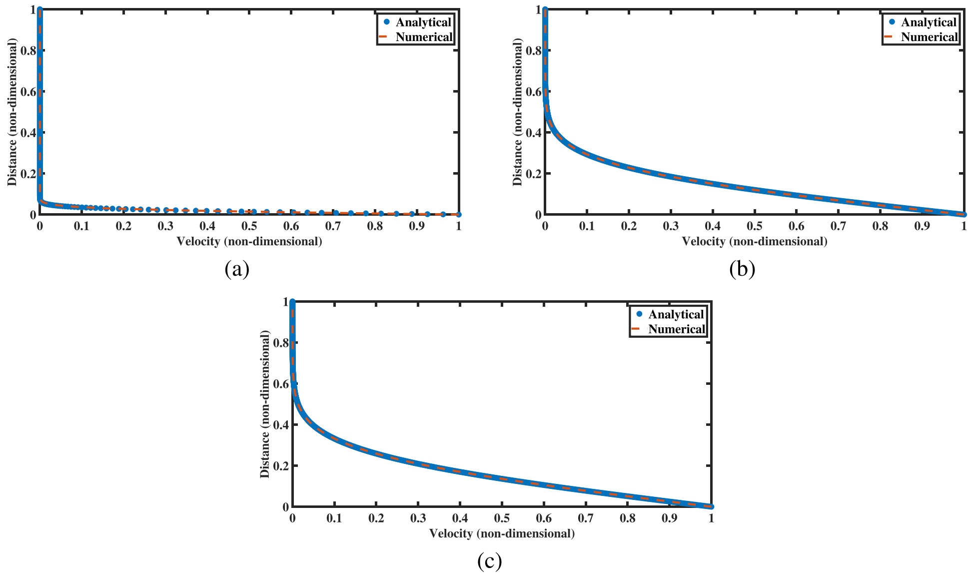

Fig. 2. Velocity profiles at $t^* = 1$ for the shear-thinning fluids between a fixed plate and an impulsively started plate: (a) Kaolin-water [2], (b) Meat extract [2] and (c) Paint [3].

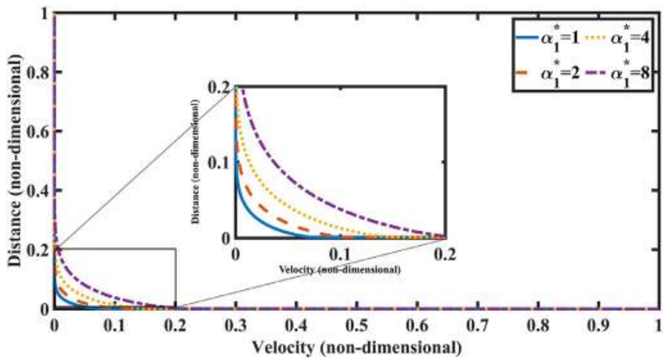

Fig. 3. Effect of variation in parameter α_1^* on the velocity v_x^* at $t^* = 1$, when $\alpha_2^* = 1$, $Re = 1000$.

α_2=1.03 $\times 10^{-1}$ s), and (for paint: α_1=1.90 $\times 10^{-1}$ N/m^2, α_2=1.07 $\times 10^2$ s) respectively. The governing PDE of the perturbation approximation is solved for each fluid's parameters, and the velocity profiles are generated. The programs are executed on an Intel-Xeon 2.30 GHz workstation (Model Fujitsu R-920). Fig. 2 provides the agreement between the exact analytical and numerical solution to equation Eqn 25 subject to the appropriate initial and boundary conditions Eqs. ((18)–(20)).

Fig. 2 (a–c) shows that at $t^* = 1$, for the fluids considered (kaolin-water[2], meat extract [2] and paint [3]), the numerical solution of the velocity matches the analytical velocity profile very well. This validates appropriateness of the numerical procedure for the same.

5.1. Parametric study

We numerically solve the full partial differential equation Eq. (24) subject to the initial condition Eq. (18) and boundary conditions, Eqs. ((19) and (20)), using pdepe solver available in MATLAB R2020a software. The governing PDE Eq. (24) is solved, and the velocity profiles are generated. The parameter study of the full numerical solution with α_1^*, α_2^*, t^* and Re is given in Figs. 3–6, respectively.

All the figures indicate very clearly the presence of a significant region adjacent to the fixed plate wherein the fluid motion is indiscernible, a feature that we would expect of a viscoplastic fluid. Figs. 3 and Fig. 4 show the effect of varying the material parameters, while Fig. 5 portrays the time evolution of the velocity profile. In all these cases, we see a large region adjacent to the fixed plate wherein the fluid is nearly static while in a very narrow region adjacent to the impulsively started plate we see the velocity being substantial. Finally, as can be seen in Fig. 6, the region wherein the fluid is nearly static becomes larger as the Reynolds number increases.

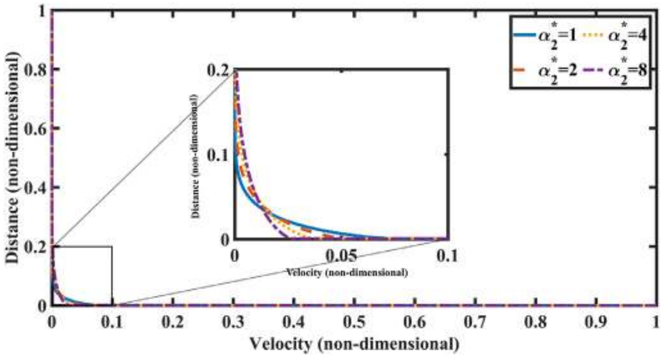

Fig. 4. Effect of variation in parameter α_2^* on the velocity v_x^* at $t^* = 1$, when $\alpha_1^* = 1$, $Re = 1000$.

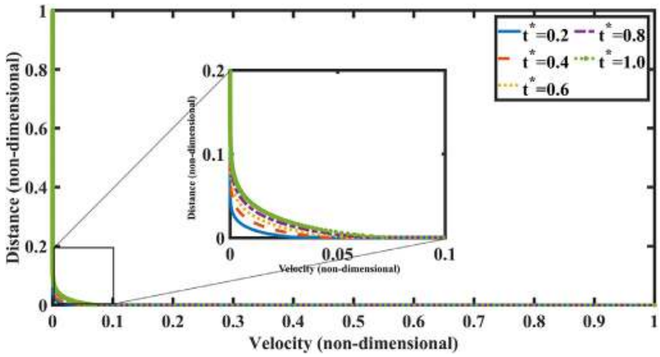

Fig. 5. Effect of variation in time t^* on the velocity v_x^*, when $\alpha_1^* = 1$, $\alpha_2^* = 1$, $Re = 1000$.

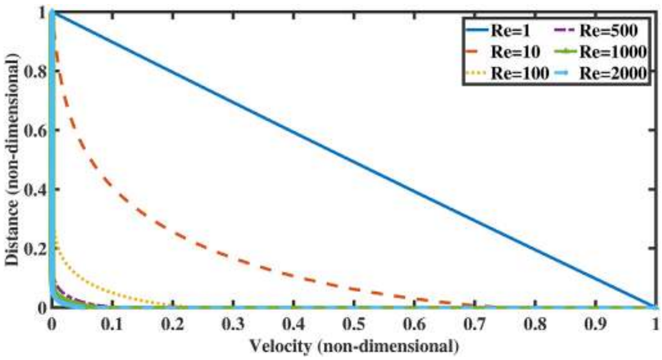

Fig. 6. Effect of variation in Reynolds number Re on the velocity v_x^*, when $\alpha_1^* = 1$, $\alpha_2^* = 1$, and at $t^* = 1$.

6. Conclusion

We have used a constitutive relation that was developed by us (see [10]) to describe the flow behaviour of the shear-thinning fluid that approximates the response of a class of viscoplastic materials to study a variant of Stokes' first problem, the flow of fluid between two infinitely long parallel plates, of which the top plate is fixed and the bottom plate is started impulsively. We used a perturbation approach to find an exact analytical solution for the velocity distribution (in space and time). The perturbation equations were solved using a numerical procedure and compared with the exact analytical solution. The numerical solution of the perturbation equations is obtained using pdepe solver and is in very good agreement with the exact analytical solution: this validates the numerical procedure. Having ascertained the efficacy of the numerical procedure we carry out a parametric study of the partial differential equation governing the flow. The effect of the dimensionless parameters, α_1^*, α_2^*, t^*, and Re on the velocity profile are analysed.

Acknowledgments

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