ON PROPERTY-(R_1) AND RELATIVE CHEBYSHEV CENTERS IN BANACH SPACES-II

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Abstract. We continue to study (strong) property-(R_1) in Banach spaces. As discussed by Pai & Nowroji in [*On restricted centers of sets*, J. Approx. Theory, **66**(2), 170–189 (1991)], this study corresponds to a triplet $(X, V, \mathcal{F})$, where X is a Banach space, V is a closed convex set, and $\mathcal{F}$ is a subfamily of closed, bounded subsets of X. It is observed that if X is a Lindenstrauss space then $(X, B_X, \mathcal{K}(X))$ has strong property-(R_1), where $\mathcal{K}(X)$ represents the compact subsets of X. It is established that for any $F \in \mathcal{K}(X)$, $\mathrm{Cent}_{B_X}(F) \neq \emptyset$. This extends the well-known fact that a compact subset of a Lindenstrauss space X admits a nonempty Chebyshev center in X. We extend our observation that Cent_{B_X} is Lipschitz continuous in $\mathcal{K}(X)$ if X is a Lindenstrauss space. If Y is a subspace of a Banach space X and $\mathcal{F}$ represents the set of all finite subsets of B_X then we observe that B_Y exhibits the condition for simultaneously strongly proximinal (viz. property-(P_1)) in X for $F \in \mathcal{F}$ if $(X, Y, \mathcal{F}(X))$ satisfies strong property-(R_1), where $\mathcal{F}(X)$ represents the set of all finite subsets of X. It is demonstrated that if P is a bi-contractive projection in ℓ_∞, then $(\ell_\infty, Range(P), \mathcal{K}(\ell_\infty))$ exhibits the strong property-(R_1), where $\mathcal{K}(\ell_\infty)$ represents the set of all compact subsets of ℓ_∞. Furthermore, stability results for these properties are derived in continuous function spaces, which are then studied for various sums in Banach spaces.

1. Introduction

1.1. **Prerequisites:** Some standard notations used in this study are introduced as follows: X indicates a Banach space, whereas a subspace denotes a closed linear subspace. For $x \in X$ and $r > 0$, $B(x, r)$ and $B[x, r]$ denote open and closed balls, respectively, each with its center at x and radius r. Furthermore, B_X and S_X denote the closed unit ball and unit sphere of X,

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respectively. Further, $\mathcal{B}(X), \mathcal{C}(X), \mathcal{K}(X)$, and $\mathcal{F}(X)$ denote the set of all closed and bounded, closed and convex, compact, and finite subsets of X, respectively. Real numbers are assumed to be the underlying field for all spaces. For $x \in X, V \in \mathcal{C}(X)$, and $B \in \mathcal{B}(X)$, the following are defined:

Notation.

(1) $r(x, B) = \sup\{\|x - b\| : b \in B\}$
(2) $\text{rad}_V(B) = \inf\{r(x, B) : x \in V\}$
(3) $\text{Cent}_V(B) = \{v \in V : r(v, B) = \text{rad}_V(B)\}$
(4) $\delta - \text{Cent}_V(B) = \{v \in V : r(v, B) \leq \text{rad}_V(B) + \delta\}$
(5) *For* $B \subseteq X$, $B_\eta = \{x \in X : d(x, B) \leq \eta\}$ for $\eta > 0$.
(6) $S_\eta(B) = \{x \in X : r(x, B) \leq \eta\}$ for $\eta > 0$.

Note that

(1) $\text{Cent}_V(B) = \left\{ \bigcap_{b \in B} B[b, \text{rad}_V(B)] \right\} \cap V.$
(2) $\delta - \text{Cent}_V(B) = \left\{ \bigcap_{b \in B} B[b, \text{rad}_V(B) + \delta] \right\} \cap V.$

However, if $V \in \mathcal{C}(X)$ and $B \in \mathcal{B}(X)$, the set $\text{Cent}_V(B)$ may be empty, although the set $\delta - \text{Cent}_V(B)$ is nonempty for any $\delta > 0$. A triplet $(X, V, \mathcal{F})$– where $V \in \mathcal{C}(X)$ and $\mathcal{F} \subseteq \mathcal{B}(X)$, a subfamily of closed bounded subsets, exhibits the *restricted center property* (**rcp**) if for all $F \in \mathcal{F}, \text{Cent}_V(F) \neq \emptyset$. Here, $\text{rad}_V(F)$ represents the radius of the smallest ball (if it exists) in X centered at V and containing F, $\text{Cent}_V(F)$ represents the possible points of the centers of these balls, and $\delta - \text{Cent}_V(F)$ represents the set of points in V which are at most $\text{rad}_V(F) + \delta$ away from F. Several researchers have investigated various characteristics (stated in the subsequent discussions) related to the entities defined above, viz. $\text{Cent}_V(F), \text{rad}_V(F)$ (see [3, 8, 10, 13]), determined by various geometric properties of the Banach space and also the type of the closed convex subset V.

Definition 1.1. [10] Let $V \in \mathcal{C}(X)$ and $\mathcal{F} \subseteq \mathcal{B}(X)$. The triplet $(X, V, \mathcal{F})$ exhibits *property-*(R_1) if for $v \in V, F \in \mathcal{F}$, and $r_1, r_2 > 0$, the conditions $r(v, F) \leq r_1 + r_2$ and $S_{r_2}(F) \cap V \neq \emptyset$ imply that $V \cap B[v, r_1 + \varepsilon] \cap S_{r_2+\varepsilon}(F) \neq \emptyset$, for all $\varepsilon > 0$.

Equivalently (see [3, Theorem 2.2, 2.4]), the triplet $(X, V, \mathcal{F})$ exhibits *property-*(R_1) if for $v \in V, F \in \mathcal{F}, r_1, r_2 > 0$ the conditions $r(v, F) < r_1 + r_2$ and $S_{r_2}(F) \cap V \neq \emptyset$ imply that $V \cap B[v, r_1] \cap S_{r_2}(F) \neq \emptyset$.

The above is a set-valued analogue of the $1\frac{1}{2}$-ball property ([14]) and clearly a subspace V has the $1\frac{1}{2}$-ball property in X if $(X, V, \mathcal{F})$ has property-(R_1) and $\mathcal{F}$ contains the singletons. The article by Pai and Nowroji ([10]) reported that if $(X, V, \mathcal{F})$ exhibits property-(R_1), then it has **rcp**. However, in [7] it is demonstrated that the $1\frac{1}{2}$-ball property is insufficient to ensure **rcp** for finite subsets. In this context we recall [12, Proposition 2.2]. It is observed for a family of bounded subsets $\mathcal{F}$, if for all $F \in \mathcal{F}$, $\mathrm{Cent}_{B_Y}(F) \neq \emptyset$ for a subspace Y, then for all $F \in \mathcal{F}$, $\mathrm{Cent}_Y(F) \neq \emptyset$. This concludes if $(X, B_X, \mathcal{F})$ exhibits **rcp**, then $\mathrm{Cent}_X(F) \neq \emptyset$ for all $F \in \mathcal{F}$.

Several characterizations for property-(R_1) have been derived by Daptari and Paul [3]. It is clear that for an arbitrary $V \in \mathcal{C}(X)$ and $v \in V$, $r(v, F) \leq \mathrm{rad}_V(F) + d(v, \mathrm{Cent}_V(F))$, for all $F \in \mathcal{B}(X)$.

Theorem 1.2. [3, Theorem 2.4] *Let V be a closed convex subset of X. Then, the triplet $(X, V, \mathcal{F})$ exhibits property-(R_1) if and only if for $v \in V$ and $F \in \mathcal{F}$, $r(v, F) = \mathrm{rad}_V(F) + d(v, \mathrm{Cent}_V(F))$.*

Daptari and Paul [4] studied a stronger version of property-(R_1), called *strong property-*(R_1), which is in fact a set-valued version of the *strong* $1\frac{1}{2}$-*ball property* (see [7]).

Definition 1.3. [4] Let $V \in \mathcal{C}(X)$ and $\mathcal{F} \subseteq \mathcal{B}(X)$. The triplet $(X, V, \mathcal{F})$ exhibits *strong property-*(R_1) if for $v \in V, F \in \mathcal{F}, r_1, r_2 > 0$ the conditions $r(v, F) \leq r_1 + r_2$ and $S_{r_2}(F) \cap V \neq \emptyset$ imply that $V \cap B[v, r_1] \cap S_{r_2}(F) \neq \emptyset$.

Several characterizations and examples of strong property-(R_1) are provided in [4]. Certain properties relevant to this study are listed below.

Theorem 1.4. *Let X be a Banach space, V be a subspace of X, and $\mathcal{F}$ be a subfamily of $\mathcal{B}(X)$. Then, the following are equivalent.*

- *(a) $(X, V, \mathcal{F})$ exhibits strong property-(R_1).*
- *(b) $(X, V, \mathcal{F})$ exhibits property-(R_1) and $\forall\ F \in \mathcal{F}$, $\mathrm{Cent}_{B_V}(F) \neq \emptyset$.*
- *(c) $\forall\ v \in V$ and $F \in \mathcal{F}$, $r(v, F) = \mathrm{rad}_V(F) + \|v - z\|$, for certain $z \in \mathrm{Cent}_V(F)$.*

In all the above characterizations for (strong) property-(R_1), one can choose $v = 0$. In [3] Daptari and Paul reported that the space $C(K)$ where K is a compact Hausdorff space yields many subspaces that satisfy (strong) property-(R_1). It is well-known that an M-ideal (see [6, pg.1]) in a Lindenstrauss space is categorized as such a subspace (see [10, Proposition 2.3] and

[4, Theorem 3.6]). For instance, if x^* is an extreme point of the dual unit ball of a Lindenstrauss space X, then $\ker(x^*)$ exhibits strong property-(R_1) for the set of all compact subsets of X. The result in [10, Proposition 2.3] follows directly in consideration of Theorem 2.4 when combined with [4, Theorem 3.5]. This study examined (strong) property-(R_1) and establishes various consequences, stability properties, and examples thereof.

1.2. **Summary of results:** The remainder of this paper is structured as follows.

Section 2 discusses several phenomena associated with property-(R_1) and strong property-(R_1). It is demonstrated with respect to the finite subsets of a Lindenstrauss space, the unit ball exhibits strong property-(R_1). Moreover, it is observed if $(X, Y, \mathcal{F}(X))$ exhibits strong property-(R_1), then $(X, B_Y, \mathcal{F}(B_X))$ has property-(P_1).

A projection $P : X \to X$ such that $\|P\| \leq 1, \|I - P\| \leq 1$ is referred to as a bi-contractive projection on X. In Section 3, the range of any bi-contractive projection in ℓ_∞ is derived as exhibiting strong property-(R_1) with respect to the compact subsets. This concludes the unit ball of such subspaces exihibits restricted Chebyshev center for all compact subsets of ℓ_∞.

Section 4 demonstrates that the properties considered in this study remain stable under continuous function spaces. For a compact Hausdorff space K, $C(K, X)$ is considered to be the vector space of all continuous functions from K that take values in X. For an $f \in C(K, X)$, $\sup_{k \in K} \|f(k)\|$ defines a norm that makes the space complete. It is demonstrated that if $(X, Y, \mathcal{F}(X))$ exhibits property-(R_1), then $(C(K, X), C(K, Y), \mathcal{F}(C(K, X)))$ has property-(R_1), and vice versa. This study adopts the technique used by Yost in [14, Theorem 2.1] to confirm that a Weierstrass–Stone subspace of $C(K, X)$ exhibits strong property-(R_1) for the finite subsets.

Section 5 discusses a few cases when (strong) property-(R_1) is stable with respect to various sums of Banach spaces.

2. Various aspects of property-(R_1)

The following theorem can be obtained from certain standard inequalities: $|\mathrm{rad}_V(F_1) - \mathrm{rad}_V(F_2)| \leq d_H(F_1, F_2)$, $|r(v_1, F) - r(v_2, F)| \leq \|v_1 - v_2\|$, for

$v_1, v_2 \in V$ and $F_1, F_2, F \in \mathcal{B}(X)$. Here, d_H represents the Hausdorff metric defined over $\mathcal{B}(X)$.

Theorem 2.1. *Let X be a Banach space and $V \in \mathcal{C}(X)$. If $(X, V, \mathcal{F}(X))$ exhibits property-(R_1), then $(X, V, \mathcal{K}(X))$ exhibits property-(R_1).*

Similar to Theorem 2.1, in certain cases, the strong property-(R_1) of a $V \in \mathcal{C}(X)$ for finite subsets is sufficient for ensuring the same for compact subsets.

Theorem 2.2. *Let X be a Banach space. Let (X, τ_X) represents a locally convex topological vector space, where τ_X be a topology weaker than $(X, \|.\|)$. In addition to that, we assume that any (norm) bounded net in X has a τ_X convergent subnet in X, and $\|.\|$ is lower semicontinuous in (X, τ_X). Then, for a τ_X-closed $V \in \mathcal{C}(X)$, $(X, V, \mathcal{K}(X))$ exhibits strong property-(R_1) whenever $(X, V, \mathcal{F}(X))$ has strong property-(R_1).*

Proof. Let $K \in \mathcal{K}(X)$. The aim is to prove that $r(0, K) = \text{rad}_V(K) + \|z\|$ for certain $z \in \text{Cent}_V(K)$.

There exists a sequence $F_n \in \mathcal{F}(X)$ such that $d_H(K, F_n) \to 0$ as $n \to \infty$. Thus, $r(0, F_n) = \text{rad}_V(F_n) + \|z_n\|$, where $z_n \in \text{Cent}_V(F_n)$ for all $n \in \mathbb{N}$. Because (z_n) is bounded in X, there exists a subnet (z_{n_i}) of (z_n) such that $z_{n_i} \to z$ in (X, τ_X) for a certain $z \in V$. Hence,

$$\|z\| \leq \liminf_i \|z_{n_i}\| = \liminf_i [r(0, F_{n_i}) - \text{rad}_V(F_{n_i})] = r(0, K) - \text{rad}_V(K).$$

CLAIM: $r(z, K) = \text{rad}_V(K)$.

Because $z_{n_i} \to z$ in (X, τ_X), the following is obtained.

$$\begin{aligned} r(z, K) &\leq \liminf_i r(z_{n_i}, K) \\ &\leq \liminf_i [r(z_{n_i}, F_{n_i}) + d(F_{n_i}, K)] \\ &= \liminf_i [\text{rad}_V(F_{n_i}) + d(F_{n_i}, K)] = \text{rad}_V(K) \end{aligned}$$

Hence, $r(z, K) = \text{rad}_V(K)$, and thus $z \in \text{Cent}_V(K)$. Therefore, $d(0, \text{Cent}_V(K)) \leq \|z\| \leq r(0, K) - \text{rad}_V(K)$. The other inequality is evident. This completes the proof. □

The following is obtained by applying Theorem 2.2.

Corollary 2.3. *Let X be a dual (reflexive) Banach space and V be a weak* (respectively, norm) closed convex subset of X. If $(X, V, \mathcal{F}(X))$ exhibits strong property-(R_1), then $(X, V, \mathcal{K}(X))$ exhibits strong property-(R_1).*

Following the arguments used in [12, Lemma 2.1] it is easy to observe that,

Theorem 2.4. *Let Y be a subspace of X and $\mathcal{F}$ be a subfamily of $\mathcal{B}(X)$. If $(X, B_Y, \mathcal{F})$ has (strong) property-(R_1) then $(X, Y, \mathcal{F})$ has (strong) property-(R_1).*

Let $V \in \mathcal{C}(X)$. Let us recall the following.

Definition 2.5. [10] For a subfamily $\mathcal{F} \subseteq \mathcal{B}(X)$, the triplet $(X, V, \mathcal{F})$ exhibits property-(P_1) if for $\varepsilon > 0$ and $F \in \mathcal{F}$ there exists $\delta(\varepsilon, F) > 0$ such that $\delta - \mathrm{Cent}_V(F) \subseteq \mathrm{Cent}_V(F) + \varepsilon B_X$.

$(X, V, \mathcal{F})$ exhibits property-(P_1) if it has property-(R_1). In addition, a Banach space X is considered a Lindenstrauss space if X^* is isometric with $L_1(\mu)$ for a certain measure space (Ω, Σ, μ). [9] is a standard reference for a comprehensive study of Lindenstrauss-type Banach spaces. Furthermore, X is a Lindenstrauss space if and only if for any finite family of closed balls $\{B_i\}_{i=1}^n$ that are pairwise intersecting in X, they actually intersect in X ([9, Theorem 5.5, 6.1]). In this statement, n can be replaced with 4. As discussed in Section 1, our next theorem extends the fact that any finite subset of a Lindenstrauss space has a nonempty Chebyshev center.

Theorem 2.6. *Let X be a Lindenstrauss space. Then, $(X, B_X, \mathcal{F}(X))$ has* **rcp**.

Proof. Let $F \in \mathcal{F}(X)$ be such that $F = \{z_1, z_2, \dots, z_n\}$. Consider the family of balls $\mathcal{B} = \{B_X, B[z_1, \mathrm{rad}_{B_X}(F)], B[z_2, \mathrm{rad}_{B_X}(F)], \dots, B[z_n, \mathrm{rad}_{B_X}(F)]\}$. Evidently, for any two $z_i, z_j \in F$, $\|z_i - z_j\| \leq 2.\mathrm{rad}_{B_X}(F)$.

Furthermore, for $z \in F$, $\|z - 0\| \leq r(0, F) \leq r(s, F) + 1$ for any $s \in B_X$.

Hence, $\|z - 0\| \leq \mathrm{rad}_{B_X}(F) + 1$ from where it follows that $B[z, \mathrm{rad}_{B_X}(F)] \cap B_X \neq \emptyset$. Because X is L_1-predual, $\bigcap_{B \in \mathcal{B}} B \neq \emptyset$.

Hence, the result follows. □

Theorem 2.7. *Let X be a Lindenstrauss space. Then, $(X, B_X, \mathcal{F}(X))$ exhibits property-(P_1).*

Proof. Let $F \in \mathcal{F}(X)$ such that $F = \{z_1, \cdots, z_n\}$ and $\varepsilon > 0$. Let $x \in \varepsilon - \mathrm{Cent}_{B_X}(F)$.

CLAIM: $x \in \mathrm{Cent}_{B_X}(F) + \varepsilon B_X$.

Our assumption yields $\|x - z_i\| \leq \text{rad}_{B_X}(F) + \varepsilon$ for all $i = 1 \cdots, n$. Thereafter, $B[x, \varepsilon] \cap B[z_i, \text{rad}_{B_X}(F)] \neq \emptyset$ for all $i = 1 \cdots, n$. Additionally, $\|z_i - z_j\| \leq 2.\text{rad}_{B_X}(F)$. Further, we have $\|z_i - 0\| \leq r(0, F) \leq r(s, F) + 1$ for all $s \in B_X$ for all $i = 1, \cdots, n$. Hence, $\|z_i - 0\| \leq \text{rad}_{B_X}(F) + 1$. Since X is a Lindenstrauss space, we have $B[x, \varepsilon] \cap (\cap_{i=1}^n B[z_i, \text{rad}_{B_X}(F)]) \cap B_X \neq \emptyset$. In other words, $B[x, \varepsilon] \cap \text{Cent}_{B_X}(F) \neq \emptyset$, and hence the claim follows. □

Remark 2.8. *(a) Let us recall that (see* [3, Theorem 2.4]*) the case $\delta = \varepsilon$ in the definition of property-(P_1) ensures property-(R_1). Hence, we have $(X, B_X, \mathcal{F}(X))$ has property-(R_1) when X is a Lindenstrauss space. In fact, we can draw a stronger conclusion, as derived in Theorem 2.9.*

(b) From [3, Theorem 2.5]*, it now follows that if X is a Lindenstrauss space, then $\text{Cent}_{B_X} : (\mathcal{F}(X), d_H) \to (\mathcal{B}(X), d_H)$ is Lipschitz continuous.*

Theorem 2.9. *Let X be a Lindenstrauss space. Then, $(X, B_X, \mathcal{F}(X))$ exhibits strong property-(R_1).*

Proof. Let $x \in B_X$ and $F = \{z_1, z_2, \ldots, z_n\} \in \mathcal{F}(X)$. Furthermore, let $r(x, F) \leq r_1 + r_2$ and $S_{r_2}(F) \cap B_X \neq \emptyset$ for $r_1, r_2 > 0$.

The condition $r(x, F) \leq r_1 + r_2$ ensures $B[x, r_1] \cap B[z_i, r_2] \neq \emptyset$ for $1 \leq i \leq n$. In addition, $S_{r_2}(F) \cap B_X \neq \emptyset$ ensures that $\|z_i - z_j\| \leq 2r_2$ for $i \neq j$ and $B[z_i, r_2] \cap B_X \neq \emptyset$. However, owing to *n.2.I.P.* of X, $B_X \cap \bigcap_{i=1}^n B[z_i, r_2] \cap B[x, r_1] \neq \emptyset$ is obtained. This completes the proof. □

As discussed in Theorem 1.4, $\text{Cent}_{B_Y}(F) \neq \emptyset$ for $F \in \mathcal{F}(X)$ if $(X, Y, \mathcal{F}(X))$ exhibits strong property-(R_1). In Theorem 2.10 it is derived that B_Y admits property-(P_1) for a suitable subfamily of finite subsets of X.

Theorem 2.10. *Let $(X, Y, \mathcal{F}(X))$ exhibit strong property-(R_1). Then, $(X, B_Y, \mathcal{F}(B_X))$ has property-(P_1).*

Proof. We utilize the techniques used in [7, Theorem 2.9]. The detailed proof is included for completeness.

Let $F \in \mathcal{F}(B_X)$, $d = \text{rad}_Y(F) > 0$, and $r(0, F) < 1 + d$. Hence, there exists $\eta > 0$ such that $r(0, F) - d = 1 - \eta$.

Considering $\varepsilon > 0$, choose $0 < \delta < 1$ such that $\delta + \frac{3\delta}{\delta + \eta} < \varepsilon$.

Now, $d = \text{rad}_{B_Y}(F)$ (see [4, Theorem 3.8]). Let $y \in \delta - \text{Cent}_{B_Y}(F)$(that is, $r(y,F) < \text{rad}_{B_Y}(F) + \delta$). Now, by strong property-(R_1), $r(y,F) = \text{rad}_Y(F) + d(y, \text{Cent}_Y(F)) = d + \inf_{z \in \text{Cent}_Y(F)} \|z - y\|$, that is, $d(y, \text{Cent}_Y(F)) = r(y,F) - d < \delta$. Hence, there exists $y_0 \in \text{Cent}_Y(F)$ such that $\|y - y_0\| < \delta$. Clearly, $\|y_0\| < \|y\| + \delta \leq 1 + \delta$.

Claim: There exists $z \in \text{Cent}_Y(F) \cap B_Y$ such that $\|y - z\| < \varepsilon$.

Now, $r(0,F) - d = 1 - \eta = d(0, \text{Cent}_Y(F))$ and there exists $z_1 \in \text{Cent}_Y(F)$ with $\|z_1\| = 1 - \eta$.

Let $w_\lambda = \lambda y_0 + (1-\lambda) z_1$, then $\|w_\lambda\| \leq \lambda(1+\delta) + (1-\lambda)(1-\eta) = 1 + \delta\lambda - (1-\lambda)\eta$.

Now, $1 + \delta\lambda - (1-\lambda)\eta = 1 \iff 1 - \lambda = \frac{\delta}{\delta+\eta} \iff \lambda = \frac{\eta}{\delta+\eta}$.

Let $\lambda = \frac{\eta}{\delta+\eta}$ and $z = w_\lambda$, then $0 < \lambda < 1$.

$\|y_0 - z\| = (1-\lambda)\|y_0 - z_1\| \leq \frac{3\delta}{\delta+\eta}$ because $\|y_0 - z_1\| \leq 2 + 1 = 3$.

Additionally, $z \in \text{Cent}_Y(F)$ because $\text{Cent}_Y(F)$ is convex and $\|z\| \leq 1 + \delta\lambda - (1-\lambda)\eta = 1$.

Hence, we have $z \in \text{Cent}_{B_Y}(F)$ and $\|y - z\| \leq \|y - y_0\| + \|y_0 - z\| < \delta + \frac{3\delta}{\delta+\eta} < \varepsilon$. This proves the claim.

Hence, $\delta - \text{Cent}_{B_Y}(F) \subseteq \text{Cent}_{B_Y}(F) + \varepsilon B_X$. □

Theorem 2.11. *Let Y be a subspace of a Banach space X. If $(X^{**}, Y^{\perp\perp}, \mathcal{F}(X^{**}))$ has property-(R_1), then $(X, Y, \mathcal{F}(X))$ has property-(R_1).*

Proof. The proof is straightforward and follows from the extended version of *Principle of local reflexivity*, as stated in [1, Theorem 3.2]. □

We do not have the answer to the following question.

Question 2.12. *Let Y be a subspace of X. Does the triplet $(X^{**}, Y^{\perp\perp}, \mathcal{F}(X^{**}))$ have property-(R_1) if $(X, Y, \mathcal{F}(X))$ has the property-(R_1)?*

3. Examples from subspaces of ℓ_∞

Let us recall that a subspace Y of X has the $1\frac{1}{2}$-ball property if and only if for $x \in X$, $\|x\| = d(x, Y) + d(0, P_Y(x))$ (see [5]). Here $P_Y(x) = \{y \in Y : \|x - y\| = d(x, Y)\}$, the set of points in Y which are nearest to x. It is well-known that in $\ell_\infty(2)$, the subspace *span*$\{(1,1)\} = Z$ (say) has the $1\frac{1}{2}$-ball property. The simplest manner of observing this is that for any

$(p,q)\in\ell_\infty(2)$, $d((p,q),Z)=\frac{|p-q|}{2}$ and the unique best approximation from (p,q) to the subspace Z is $(\frac{p+q}{2},\frac{p+q}{2})$. Hence for $(p,q)\in\ell_\infty(2)$, the above identity turns out to be equivalent to $|p|\vee|q|=|\frac{p-q}{2}|+|\frac{p+q}{2}|$, which is true for an arbitrary pair of real numbers p,q.

Consequently, using similar arguments, it can be easily observed that $span\{(1,-1)\}$ (=W say) has the $1\frac{1}{2}$-ball property in $\ell_\infty(2)$. This study claims that both subspaces, viz. Z,W, exhibit property-(R_1) for finite subsets of $\ell_\infty(2)$.

Proposition 3.1. *Let* $X=\ell_\infty(2)$ *and* $Z=span\{(1,1)\}$. *Then,* $(X,Z,\mathcal{F}(X))$ *exhibits strong property-*(R_1).

Proof. Let $F\in\mathcal{F}(X)$. Furthermore, let $(z,z)\in Z$ and $r_1,r_2>0$ be such that $r((z,z),F)\le r_1+r_2$ and $S_{r_2}(F)\cap Z\neq\emptyset$.

Claim: $B[(z,z),r_1]\cap S_{r_2}(F)\cap Z\neq\emptyset$.

If $card(F)=1$, then the claim follows from the fact that Z has the $1\frac{1}{2}$-ball property in X. Suppose that the assertion holds for $card(F)=n$. The following proof is when $card(F)=n+1$. Assume that $F=\{(x_1,y_1),(x_2,y_2),\ldots,(x_{n+1},y_{n+1})\}$.

Let $(p,p)\in S_{r_2}(F)\cap Z=\cap_{i=1}^{n+1}B[(x_i,y_i),r_2]\cap Z$. Because Z exhibits strong property-(R_1) when F has n elements, the following is obtained for certain $(s_i,s_i)\in\ell_\infty(2)$ for $i=1,2,\ldots,n$: $(s_i,s_i)\in B[(z,z),r_1]\cap\cap_{\substack{j=1\\ j\neq i}}^{n+1}B[(x_j,y_j),r_2]$. Here, s_1,s_2 are chosen and the arguments are as stated below.

Case 1: When $p\le s_1\le s_2$.

Then, $-r_2\le p-x_1\le s_1-x_1\le s_2-x_1\le r_2$ and $-r_2\le p-y_1\le s_1-y_1\le s_2-y_1\le r_2$, and thus $(s_1,s_1)\in B[(x_1,y_1),r_2]$.

Thus, $(s_1,s_1)\in B[(z,z),r_1]\cap\bigcap_{i=1}^{n+1}B[(x_i,y_i),r_2]\cap Z=B[(z,z),r_1]\cap S_{r_2}(F)\cap Z$.

Similar arguments can be used for the cases:

Case 2: $p\le s_2\le s_1$,

Case 3: $s_1\le s_2\le p$ and

Case 4: $s_2\le s_1\le p$.

We now remain with the following cases.

Case 5: When $s_1\le p\le s_2$.

Then, $z\le s_1+r_1\le p+r_1$ and $p-r_1\le s_2-r_1\le z$, and thus $|p-z|\le r_1$.

Thus, $(p,p)\in B[(z,z),r_1]\cap\bigcap_{i=1}^{n+1}B[(x_i,y_i),r_2]\cap Z=B[(z,z),r_1]\cap S_{r_2}(F)\cap Z$.

CASE 6: When $s_2 \leq p \leq s_1$.

Then, $z \leq s_2 + r_1 \leq p + r_1$ and $p - r_1 \leq s_1 - r_1 \leq z$, and thus $|p - z| \leq r_1$.

Thus, $(p, p) \in B[(z, z), r_1] \cap \bigcap_{i=1}^{n+1} B[(x_i, y_i), r_2] \cap Z = B[(z, z), r_1] \cap S_{r_2}(F) \cap Z$.

Hence, the result follows when $card(F) = n + 1$, and this completes the proof. □

Moreover, a similar conclusion can be derived for the subspace $span\{(1, -1)\}$.

Proposition 3.2. *Let $X = \ell_\infty(2)$ and $W = span\{(1, -1)\}$. Then, $(X, W, \mathcal{F}(X))$ has strong property-(R_1) in X.*

We now establish that the range of any bi-contractive projection in ℓ_∞ exhibits property-(R_1). Let us recall that, if $P : \ell_\infty \to \ell_\infty$ is a bi-contractive projection, then either $Px(n) = \frac{x(n)+x(\tau(n))}{2}$ or $Px(n) = \frac{x(n)-x(\tau(n))}{2}$ for $x \in \ell_\infty$ and $n \in \mathbb{N}$ (see [2, Theorem 3.9]). Here, $\tau : \mathbb{N} \to \mathbb{N}$ is a permutation such that $\tau^2 = I$ (identity).

Thus, if P is a bi-contractive projection on ℓ_∞ and τ is the corresponding permutation on $\mathbb{N}$, then, for $n \in \mathbb{N}$, either $\tau(n) = n$ or $\tau(n) = m$ for certain $m \in \mathbb{N}$ and, in the second case, $\tau(m) = n$. This concludes for a bi-contractive projection $P : \ell_\infty \to \ell_\infty$,

(1) When $\tau(n) = n$: either $Px(n) = x(n)$ or $Px(n) = 0$.
(2) When $\tau(n) = m$ $(n \neq m)$: either $(Px(n), Px(m)) = \alpha(1, 1)$, or $(Px(n), Px(m)) = \alpha(1, -1)$ for some scalar α.

Hence, if $P \neq 0$, then $P(\ell_\infty)$ is isometrically isomorphic with either $(\Pi_{n\in A}(\alpha_n, \alpha_n))_\infty \oplus_\infty (\Pi_{m\in B}(\beta_m, -\beta_m))_\infty \oplus_\infty W$ or ℓ_∞. Here, W is an M-summand of ℓ_∞ and $A, B \subseteq \mathbb{N}$. Based on [4, Theorem 3.5], W exhibits strong property-(R_1) in ℓ_∞. Furthermore, the subspaces $(\Pi_{n\in A}(\alpha_n, \alpha_n))_\infty$ and $(\Pi_{m\in B}(\beta_m, -\beta_m))_\infty$ exhibit strong property-(R_1) in ℓ_∞ when considering Propositions 3.1 and 3.2 and Theorem **??**. Finally, according to Theorem **??**, the subspace $range(P)$ has strong property-(R_1) in ℓ_∞. Hence, the following is obtained.

Theorem 3.3. *Let P be a bi-contractive projection in ℓ_∞ and $Y = range(P)$. Then, $(\ell_\infty, Y, \mathcal{F}(\ell_\infty))$ has strong property-(R_1).*

We now consider the derivation for the subspace $span\{(1, 1, \ldots, 1, \ldots)\}$, which exhibits strong property-(R_1) in ℓ_∞ for finite subsets.

Theorem 3.4. *Let $X = \ell_\infty$ and $Y = span\{(1, 1, \ldots)\}$. Then, $(X, Y, \mathcal{F}(X))$ exhibits strong property-(R_1).*

Proof. Let $F \in \mathcal{F}(X)$, $(x, x, \ldots) \in Y$, $r_1, r_2 > 0$ be such that $r((x, x, \ldots), F) \leq r_1 + r_2$ and $S_{r_2}(F) \cap Y \neq \emptyset$.

Claim: $B[(x, x, \ldots), r_1] \cap S_{r_2}(F) \cap Y \neq \emptyset$.

Step 1: Let $card(F) = 1$. Let $F = \{(x(1), x(2), \ldots)\}$. Then, we have $B[(x, x, \ldots), r_1] \cap B[(x(1), x(2), \ldots), r_2] \neq \emptyset$ and $B[(x(1), x(2), \ldots), r_2] \cap Y \neq \emptyset$. Let $(z(1), z(2), \ldots) \in B[(x, x, \ldots), r_1] \cap B[(x(1), x(2), \ldots), r_2]$ and $(y, y, \ldots) \in B[(x(1), x(2), \ldots), r_2]$.

Let $\alpha = \inf\limits_{i \in \mathbb{N}} z(i)$ and $\beta = \sup\limits_{i \in \mathbb{N}} z(i)$. Then, $\alpha \leq z(i) \leq \beta$ for all $i \in \mathbb{N}$.

Case 1: When $y \leq \alpha$.

Then, $y \leq \alpha \leq z(i)$ for all $i \in \mathbb{N}$. Then, $-r_2 \leq y - x(i) \leq \alpha - x(i) \leq z(i) - x(i) \leq r_2$ for all $i \in \mathbb{N}$. Hence, $|\alpha - x(i)| \leq r_2$ for all $i \in \mathbb{N}$. Let $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that $z(N) - \varepsilon \leq \alpha$. Hence, $x - \varepsilon \leq z(N) + r_1 - \varepsilon \leq \alpha + r_1$ and $\alpha - r_1 \leq z(N) - r_1 \leq x$. Thus, $|\alpha - x| \leq r_1 + \varepsilon$.

Therefore, $(\alpha, \alpha, \ldots) \in B[(x, x, \ldots), r_1] \cap B[(x(1), x(2), \ldots), r_2]$.

Case 2: When $\alpha \leq y \leq \beta$.

Let $\varepsilon > 0$. There exists $N, N' \in \mathbb{N}$ such that $z(N) - \varepsilon \leq \alpha$ and $\beta \leq z(N') + \varepsilon$. Subsequently, $x - \varepsilon \leq z(N) - \varepsilon + r_1 \leq \alpha + r_1 \leq y + r_1$ and $y - r_1 \leq \beta - r_1 \leq z(N') + \varepsilon - r_1 \leq x + \varepsilon$. Then, $|y - x| \leq r_1 + \varepsilon$.

Hence, $(y, y, \ldots) \in B[(x, x, \ldots), r_1] \cap B[(x(1), x(2), \ldots), r_2]$.

Case 3: When $\beta \leq y$.

Then, $z(i) \leq \beta \leq y$ for all $i \in \mathbb{N}$. Furthermore, $-r_2 \leq z(i) - x(i) \leq \beta - x(i) \leq y - x(i) \leq r_2$ for all $i \in \mathbb{N}$. Hence, $|\beta - x(i)| \leq r_2$ for all $i \in \mathbb{N}$. Let $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that $\beta \leq z(N) + \varepsilon$. Hence, $x \leq z(N) + r_1 \leq \beta + r_1$ and $\beta - r_1 \leq z(N) - r_1 + \varepsilon \leq x + \varepsilon$. Thus, $|\beta - x| \leq r_1 + \varepsilon$.

Thus, $(\beta, \beta, \ldots) \in B[(x, x, \ldots), r_1] \cap B[(x(1), x(2), \ldots), r_2]$.

Hence, $(X, Y, \mathcal{F}(X))$ has strong property-(R_1) when $card(F) = 1$.

Step 2: Suppose that the assertion holds for all $F \in \mathcal{F}(X)$ when $card(F) = n$. Let $card(F) = n + 1$. Let $F = \{x_1, \ldots, x_{n+1}\}$, where $x_i = (x_i(1), x_i(2), \ldots)$ for $i = 1, \ldots, n + 1$. Then, we have $B[(x, x, \ldots), r_1] \cap B[(x_i(1), x_i(2), \ldots), r_2] \neq \emptyset$ for all $i = 1, \ldots, n + 1$ and $\cap_{i=1}^{n+1} B[(x_i(1), x_i(2), \ldots), r_2] \cap Y \neq \emptyset$. Now, because $(X, Y, \mathcal{F}(X))$

exhibits property-(R_1) when $card(F) = n$, the following is obtained: $B[(x,x,\ldots),r_1] \cap \cap_{\substack{i=1\\i\neq 2}}^{n+1} B[(x_i(1),x_i(2),\ldots),r_2] \cap Y \neq \emptyset$ and $B[(x,x,\ldots),r_1] \cap \cap_{i=2}^{n+1} B[(x_i(1),x_i(2),\ldots),r_2] \cap Y \neq \emptyset$. Let $(p,p,\ldots) \in B[(x,x,\ldots),r_1] \cap \cap_{\substack{i=1\\i\neq 2}}^{n+1} B[(x_i(1),x_i(2)\ldots),r_2]$, $(q,q,\ldots) \in B[(x,x,\ldots),r_1] \cap \cap_{i=2}^{n+1} B[(x_i(1),x_i(2),\ldots),r_2]$ and $(s,s,\ldots) \in \cap_{i=1}^{n+1} B[(x_i(1),x_i(2),\ldots),r_2]$.

Case 1: When $s \leq p \leq q$.

Then, $-r_2 \leq s - x_2(i) \leq p - x_2(i) \leq q - x_2(i) \leq r_2$ for all $i \in \mathbb{N}$. Thus, $(p,p,\ldots) \in B[(x_2(1),x_2(2),\ldots),r_2]$.

Hence, $(p,p,\ldots) \in B[(x,x,\ldots),r_1] \cap \cap_{i=1}^{n+1} B[(x_i(1),x_i(2),\ldots),r_2]$.

Similar ideas can be adopted to establish the following cases.

Case 2: $s \leq q \leq p$.

Case 3: $p \leq q \leq s$.

Case 4: $q \leq p \leq s$.

We now remain with the following cases.

Case 5: $p \leq s \leq q$.

Then, $x \leq p + r_1 \leq s + r_1$ and $s - r_1 \leq q - r_1 \leq x$. Thus, $|s - x| \leq r_1$.

Hence, $(s,s,\ldots) \in B[(x,x,\ldots),r_1] \cap \cap_{i=1}^{n+1} B[(x_i(1),x_i(2),\ldots),r_2]$.

Case 6: When $q \leq s \leq p$.

Then, $x \leq q + r_1 \leq s + r_1$ and $s - r_1 \leq p - r_1 \leq x$. Thus, $|s - x| \leq r_1$.

Hence, $(s,s,\ldots) \in B[(x,x,\ldots),r_1] \cap \cap_{i=1}^{n+1} B[(x_i(1),x_i(2),\ldots),r_2]$.

Thus, the assertion holds for all $F \in \mathcal{F}(X)$. □

It is clear that the subspaces of type $(\Pi_n(\alpha_n,\alpha_n))_\infty$ or $(\Pi_m(\beta_m,-\beta_m))_\infty$ stated before Theorem 3.3 are w^*-closed, and so is the subspace in Theorem 3.4. Hence, by Corollary 2.3, the conclusions in Theorems 3.3 and 3.4 remain valid for $\mathcal{K}(X)$.

4. Subspaces of $C(K,X)$ with property-(R_1)

In this section, the following fact is proven. By K and $C(K,X)$, we denote a compact Hausdorff space and the Banach space of X-valued continuous functions over K, as discussed in Section 1.

Theorem 4.1. *Let Y be a subspace of X. Then, $(X,Y,\mathcal{F}(X))$ has property-(R_1) if and only if $(C(K,X),C(K,Y),\mathcal{F}(C(K,X)))$ has property-(R_1).*

Before proving Theorem 4.1, a few supporting results must be derived. For a real valued function $f : S \to \mathbb{R}$, we denote $S(f) = \overline{\{t \in S : f(t) \neq 0\}}$, the support of f.

Proposition 4.2. *Let $f_1, f_2, \ldots, f_k \in C(K)$ and $\varepsilon > 0$. Then, there exists a finite family $(\varphi_i)_{i=1}^m \subseteq C(K)$, where $(\varphi_i)_{i=1}^m$ forms a partition of unity and there exists $h_1, h_2, \ldots, h_k \in span\{\varphi_i : 1 \leq i \leq m\}$ such that $\|f_i - h_i\|_\infty < \varepsilon$ for $1 \leq i \leq k$.*

Proof. CASE 1: When $k = 1$.

Let $\{V_i : 1 \leq i \leq n\}$ be a finite open cover of K such that $|f(z) - f(w)| < \varepsilon$ for $z, w \in V_i$ and $1 \leq i \leq n$. Let $(\varphi_i)_{i=1}^n$ be a partition of unity such that $0 \leq \varphi_i \leq 1$, $1 \leq i \leq n$ and $S(\varphi_i) \subseteq V_i$ ([11, Theorem 2.13]). Choose $v_i \in V_i$ and define $h = \sum_{i=1}^n f(v_i)\varphi_i$. Then, $|f(x) - h(x)| = |\sum_{i=1}^n f(x)\varphi_i(x) - \sum_{i=1}^n f(v_i)\varphi_i(x)| \leq \sum_j |\varphi_{i_j}(x)||f(x) - f(v_{i_j})|$. The last sum is taken over all those j's for which $x \in V_{i_j}$. Clearly, $\sum_j \varphi_{i_j}(x)|f(x) - f(v_{i_j})| \leq \varepsilon$.

CASE 2: When $k > 1$.

Without loss of generality, this study assumes that $k = 2$; no new ideas are involved in other values of k.

Let $(\varphi_i)_{i=1}^n$ and $(\varrho_i)_{i=1}^m$ be two partitions of unity in $C(K)$, such that there exists h_1, h_2, where $h_1 \in span\{\varphi_i : 1 \leq i \leq n\}$ and $h_2 \in span\{\varrho_i : 1 \leq i \leq m\}$, where $\|f_i - h_i\| < \varepsilon$ for $i = 1, 2$. Then, $\{\varphi_i\varrho_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a partition of unity and $h_i \in span\{\varphi_i\varrho_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. This completes the proof. ☐

Let $(\varphi_i)_{i=1}^N$ be a finite partition of unity in $C(K)$ corresponding to an open cover $\mathcal{U}$ of K obtained as in [11, Theorem 2.13]. Then, we call $(\varphi_i)_{i=1}^N$ a partition of unity in $C(K)$ subordinate to the cover $\mathcal{U}$. In this case, $(\varphi_i)_{i=1}^N$ corresponds to a subspace Z of $C(K, X)$: $Z = \{\sum_{i=1}^N x_i\varphi_i : x_i \in X, 1 \leq i \leq n\}$. It is clear that $Z \cong \bigoplus_{\ell_\infty(N)} X$.

Proposition 4.3. *Let X be a Banach space, K be a compact Hausdorff space, and $f_1, f_2, \ldots, f_n \in C(K, X)$. Then, for $\varepsilon > 0$, there exists a subspace Z of $C(K, X)$, where $Z \cong \bigoplus_{\ell_\infty(m)} X$ for some m, and $d(f_i, Z) \leq \varepsilon$, $1 \leq i \leq n$.*

Proof. This study followed Proposition 4.2 to construct a subspace $Z \subseteq C(K, X)$. If $S_i = f_i(K)$, then $S_i \subseteq X$ is a compact set. Let us fix i. For every $s \in S_i$, let $B(s, \varepsilon) \cap S_i$ be a ball in S_i. For a finite sub-cover $\mathcal{U}_i$

of $\{f_i^{-1}(B(s,\varepsilon)\cap S_i) : s\in S_i\}$, we may choose a finite partition of unity subordinate to the cover $\mathcal{U}_i$; say, (φ_j). If $(s_j)_{j=1}^n \subseteq S_i$ is a finite set of points corresponding to the cover $\mathcal{U}_i$, then $\|f_i - \sum_j \varphi_j s_j\| \leq \varepsilon$.

We now continue the process for other values of i. Following the arguments for n functions as derived in Proposition 4.2, it is concluded for a finite dimensional subspace Z of $C(K,X)$, where $Z \cong \bigoplus_{\ell_\infty(m)} X$, for some $m\in\mathbb{N}$. Clearly, $d(f_i,Z)\leq\varepsilon$, $1\leq i\leq n$, and this completes the proof. $\square$

We state the following result without proof, which is useful to derive Theorem 4.5. A routine verification of the (strong) property-(R_1) can lead to the proof of the following. We derive similar results in Section 5. For any unexplained notation used in Theorem 4.4, we refer to section 5.

Theorem 4.4. *Let Y be a subspace of X. Then, $(X,Y,\mathcal{B}(X))$ exhibits (strong) property-(R_1) if and only if $(X_\infty, Y_\infty, \mathcal{B}(X_\infty))$ exhibits (strong) property-(R_1).*

We are now ready to prove Theorem 4.1.

Theorem 4.5. *Let Y be a subspace of X. Then, $(X,Y,\mathcal{F}(X))$ has property-(R_1) if and only if $(C(K,X), C(K,Y), \mathcal{F}(C(K,X)))$ has property-(R_1).*

Proof. Let $F\in\mathcal{F}(C(K,X))$ and $g\in C(K,Y)$. Let $r_1,r_2>0$ such that $S_{r_2}(F)\cap C(K,Y)\neq\emptyset$ and $r(g,F)\leq r_1+r_2$. Consequently, the following is developed.

Claim: $S_{r_2+\varepsilon}(F)\cap B(g,r_1+\varepsilon)\cap C(K,Y)\neq\emptyset$ for all $\varepsilon>0$.

Suppose that $h\in S_{r_2}(F)\cap C(K,Y)$. From Proposition 4.3, there exists $Z\subseteq C(K,X)$, where $Z\cong\bigoplus_{\ell_\infty(k)} X$ for certain k such that for all $f\in F$, $d(f,Z)<\varepsilon$. Let $F=\{f_1,f_2,\ldots,f_n\}$ and $F'=\{f_1',f_2',\ldots,f_n'\}$ where $f_i'\in Z$ and $\|f_i-f_i'\|<\varepsilon$. Additionally, there exists $W\subseteq C(K,Y)$, where $W\cong\bigoplus_{\ell_\infty(m)} Y$ such that there exist $g',h'\in W$ and $\|g-g'\|<\varepsilon$, $\|h-h'\|<\varepsilon$, here g and h are taken as above. Subsequently, without loss of generality, $m=k$ may be assumed. Furthermore, from the assumption, $S_{r_2+2\varepsilon}(F')\cap W\neq\emptyset$ and $r(g',F')\leq r_1+r_2+2\varepsilon$ is obtained. In addition, from Theorem 4.4 $(\bigoplus_{\ell_\infty(k)} X, \bigoplus_{\ell_\infty(k)} Y, \mathcal{F}(\bigoplus_{\ell_\infty(k)} X))$ with property-(R_1) is obtained; hence, $S_{r_2+2\varepsilon}(F')\cap B(g',r_1+2\varepsilon)\cap W\neq\emptyset$.

Let $h\in S_{r_2+2\varepsilon}(F')\cap B(g',r_1+2\varepsilon)\cap W$. After identifying h with an element in $C(K,Y)$, we obtain $h\in S_{r_2+3\varepsilon}(F)\cap B(g,r_1+3\varepsilon)\cap C(K,Y)$. Moreover, because $\varepsilon>0$ is arbitrary, the proof follows. $\square$

However, it is not yet known whether analogous results such as Theorem 4.5 are true for the spaces of the form $L_1(\mu, X)$ or not. Nevertheless, if the triplet $(L_1(\mu, X), L_1(\mu, Y), \mathcal{F}(L_1(\mu, X)))$ has property-(R_1), then $(X, Y, \mathcal{F}(X))$ has property-(R_1).

Theorem 4.6. *Let E be a real Lindenstrauss space, K and S be compact Hausdorff spaces, and $\psi : K \to S$ be a continuous onto map. Let $\psi^* : C(S, E) \to C(K, E)$ be the natural isometric embedding expressed as $\psi^* f = f \circ \psi$. Then, $(C(K, E), \psi^* C(S, E), \mathcal{F}(C(K, E)))$ exhibits strong property-(R_1).*

Proof. Let $F \in \mathcal{F}(C(K, E))$ such that $F = \{f_1, \dots, f_n\}$. Suppose that $r > 0$ such that $r(0, F) \leq 1 + r$ and $S_r(F) \cap \psi^* C(S, E) \neq \emptyset$. Furthermore, define $\eta : S \to \mathcal{B}(E)$ by

$$\begin{aligned} \eta(y) &= B[0,1] \cap \left(\bigcap_{k \in \psi^{-1}(y)} \cap_{i=1}^n B[f_i(k), r] \right) \\ &= B[0,1] \cap \left(\cap_{i=1}^n \{a \in E : f_i(\psi^{-1}(y)) \subseteq B[a, r]\} \right). \end{aligned}$$

Each $\eta(y)$ is closed and convex.

Claim: $\eta(y) \neq \emptyset$ for all $y \in S$.

Let $\psi^* g \in \psi^* C(S, E) \cap S_r(F) = \psi^* C(S, E) \cap (\cap_{i=1}^n B[f_i, r])$.

For $k_1, k_2 \in \psi^{-1}(y)$, we have

$$\begin{aligned} \|f_i(k_1) - f_j(k_2)\| &\leq \|f_i(k_1) - g(y)\| + \|f_j(k_2) - g(y)\| \\ &\leq \|f_i - \psi^* g\| + \|f_j - \psi^* g\| \leq 2r \end{aligned}$$

for $i, j = 1, \dots, n$. Hence, $B[f_i(k_1), r] \cap B[f_j(k_2), r] \neq \emptyset$. Because $\|f_i\| \leq r + 1$, $B[0,1] \cap B[f_i(k), r] \neq \emptyset$ for all $i = 1, \dots, n$. Thus, the entire family of balls defining $\eta(y)$ has a pairwise non-empty intersection property. Moreover, owing to the collection of centers $\{0\} \cup \cup_{i=1}^n f_i(\psi^{-1}(y))$ being compact, $\eta(y) \neq \emptyset$ is obtained.

Claim: η is lower semicontinuous.

Let $G \subseteq E$ be open, and $y_0 \in \{y : \eta(y) \cap G \neq \emptyset\}$ and $a \in \eta(y_0) \cap G$. Then, $\|a\| \leq 1, f_i(\psi^{-1}(y_0)) \subseteq B[a, r]$ and $B[a, \varepsilon] \subseteq G$ for certain $\varepsilon > 0$ and for all $i = 1, \dots, n$. As K is compact, the map $y \to \psi^{-1}(y)$ is upper semicontinuous. Hence, $N = \cap_{i=1}^n \{y : f_i(\psi^{-1}(y)) \subseteq \operatorname{int} B[a, r+\varepsilon]\}$ is an open set containing y_0. If $y \in N$, then $B[a, \varepsilon] \cap B[f_i(k), r] \neq \emptyset$ for all $k \in \psi^{-1}(y)$ and for all $i = 1, \dots, n$. Additionally, $B[a, \varepsilon] \cap B[0, 1] \neq \emptyset$. Furthermore, because E is a real Lindenstrauss space, $\eta(y) \cap B[a, \varepsilon] \neq \emptyset$ for all $y \in N$. Thus, $N \subseteq \{y : \eta(y) \cap G \neq \emptyset\}$, where $\{y : \eta(y) \cap G \neq \emptyset\}$ is open and η is lower semicontinuous.

Now, applying Michael's selection theorem, a continuous selection $h : S \to E$ is obtained, such that $h(y) \in \eta(y)$ for all $y \in S$. Accordingly, $\psi^* h \in \cap_{i=1}^n B[f_i, r] \cap B[0, 1] \cap \psi^* C(S, E)$. This completes the proof. □

The following is obtained as a consequence of Theorem 4.6.

Corollary 4.7. *Let K, S, E, ψ be as in Theorem 4.6. Furthermore, $y_0 \in S$ is set and let $M = \{\psi^* f : f \in C(S, E)$ and $f(y_0) = 0\}$. Then, $(C(K, E), M, \mathcal{F}(C(K, E)))$ has strong property-(R_1).*

Proof. Let f, r, η be similar to that in the proof of Theorem 4.6. If $\psi^* g \in M \cap \cap_{i=1}^n B[f_i, r]$, $\|f_i(x)\| = \|f_i(x) - \psi^* g(x)\| \leq r$ for all $x \in \psi^{-1}(y_0)$ and for all $i = 1, \ldots, n$. Hence, $0 \in \eta(y_0)$. If we define $\eta_0 : S \to \mathcal{B}(E)$ by

$$\eta_0(y) = \begin{cases} \eta(y) & \text{if } y \neq y_0 \\ \{0\} & \text{if } y = y_0 \end{cases}$$

This η_0 is clearly lower semicontinuous. Subsequently, on applying Michael's selection theorem, a continuous selection $h : S \to E$ is obtained, such that $h(y) \in \eta_0(y)$ for all $y \in K$. Hence, the assertion follows. □

5. Stability results

For a Banach space X we introduce the following notations.

$X_0 = \bigoplus_{c_0} X = \{(x_n) : x_n \in X, \lim_n \|x_n\| = 0\}$,

$X_\infty = \bigoplus_{\ell_\infty} X = \{(x_n) : x_n \in X, \sup_n \|x_n\| < \infty\}$

$X_1 = \bigoplus_{\ell_1} X = \{(x_n) : x_n \in X, \sum_n \|x_n\| < \infty\}$

$\bigoplus_{\ell_1(m)} X = \oplus_{i=1}^m X$ with norm $\sum_{i=1}^m \|x_i\|$ and

$\bigoplus_{\ell_\infty(m)} X = \oplus_{i=1}^m X$ with norm $\max_{i=1}^m \|x_i\|$.

Theorem 5.1. *Let Y be a subspace of X. Then, $(X, Y, \mathcal{F}(X))$ has (strong) property-(R_1) if and only if $(X_0, Y_0, \mathcal{F}(X_0))$ has (strong) property-(R_1).*

Proof. First, the result for property-(R_1) is derived.

It is sufficient to prove that $(X_0, Y_0, \mathcal{F}(X_0))$ exhibits property-(R_1) if $(X, Y, \mathcal{F}(X))$ has property-(R_1). Proof of this fact is outlined below.

Let $F \in \mathcal{F}(X_0)$ and $F(n) \subseteq X$ be the corresponding component, that is, $F(n) \in \mathcal{F}(X)$. Suppose $F = \{x_1, \dots, x_k\}$, $y_0 = (y_0(n)) \in Y_0$ and $r_1, r_2 > 0$ be such that $S_{r_2}(F) \cap Y_0 \neq \emptyset$ and $r(y_0, F) < r_1 + r_2$.

CLAIM: $S_{r_2}(F) \cap B[y_0, r_1] \cap Y_0 \neq \emptyset$

It is clear that there exists N sufficiently large, such that $0 \in S_{r_2}(F(n)) \cap B[y_0(n), r_1] \cap Y$, for all $n > N$. Now, for $1 \leq n \leq N$, we choose $y(n) \in S_{r_2}(F(n)) \cap B[y_0(n), r_1] \cap Y$. As a result, an element $(y(n))$, a member of the set as specified in the claim above, exists.

Nevertheless, regarding the remaining part, it is sufficient to prove that $(X_0, Y_0, \mathcal{F}(X_0))$ exhibits strong property-(R_1) if $(X, Y, \mathcal{F}(X))$ exhibits strong property-(R_1).

This study only shows that $\text{Cent}_{B_{Y_0}}(F) \neq \emptyset$, for $F \in \mathcal{F}(X_0)$.

Let $F \in \mathcal{F}(X_0)$ and $F(n) \in \mathcal{F}(X)$ be as defined above. Based on the assumption, $\text{Cent}_{B_Y}(F(n)) \neq \emptyset$ for all $n \in \mathbb{N}$. Let $y(n) \in \text{Cent}_{B_Y}(F(n))$ be obtained for all $n \in \mathbb{N}$.

It is clear that $y = (y(n)) \in \bigoplus_{c_0} Y$, because $\text{rad}_{B_Y}(F(n)) \to 0$.

Since for any $z = (z(1), z(2), \cdots) \in \bigoplus_{c_0} B_Y$, $r(y, F) \leq r(z, F)$ and because $\|(y(n))\|_\infty \leq 1$, $y \in \text{Cent}_{\bigoplus_{c_0} B_Y}(F) = \text{Cent}_{B_{Y_0}}(F)$, the result follows from Theorem 1.4. $\square$

Remark 5.2. *(a) If Y is a finite co-dimensional proximinal subspace of c_0, then there exists $n \in \mathbb{N}$, such that $Y = F \oplus_\infty Z$, where F is a subspace of $\ell_\infty(n)$ and $Z = \{(x_i) \in c_0 : x_i = 0, 1 \leq i \leq n\}$. From Theorem 4.4, it is clear that Y has property-(R_1) in c_0 if and only if F has property-(R_1) in $\ell_\infty(n)$. We do not know the characterization of $\alpha_i \in \ell_1(n)$ where $1 \leq i \leq m$, considering dim$(c_0/Y) = m$, satisfying the condition that $\cap_i \ker(\alpha_i)$ has property-(R_1) in $\ell_\infty(n)$.*

(b) If Y is a finite co-dimensional proximinal subspace of c_0, then from the decomposition $Y = F \oplus_\infty Z$, it is also clear that Y has property-(R_1) in c_0 if and only if Y has property-(R_1) in ℓ_∞.

We now consider the result for the ℓ_1-sum.

Proposition 5.3. *Let X be a Banach space and Y_1, Y_2 be two subspaces of X. Then, for $F_1, F_2 \in \mathcal{B}(X)$ the following can be obtained:*

(a) $r((y_1, y_2), F_1 \times F_2) = r(y_1, F_1) + r(y_2, F_2) \ \forall \ (y_1, y_2) \in Y_1 \oplus_1 Y_2$.

(b) $\text{rad}_{Y_1 \oplus_1 Y_2}(F_1 \times F_2) = \text{rad}_{Y_1}(F_1) + \text{rad}_{Y_2}(F_2)$.

(c) $\text{Cent}_{Y_1 \oplus_1 Y_2}(F_1 \times F_2) = \text{Cent}_{Y_1}(F_1) \oplus_1 \text{Cent}_{Y_2}(F_2)$.

(d) $d(0, \mathrm{Cent}_{Y_1\oplus_1 Y_2}(F_1 \times F_2)) = d(0, \mathrm{Cent}_{Y_1}(F_1)) + d(0, \mathrm{Cent}_{Y_2}(F_2))$.

Let X be a Banach space. For a fixed n, define $\mathcal{H} = \{\Pi_{i=1}^n F_i : F_i \in \mathcal{F}(X)\}$. The following is obtained as a consequence of Proposition 5.3.

Theorem 5.4. *Let X be a Banach space and Y be a subspace of X. Then, $(X, Y, \mathcal{F}(X))$ exhibits property-(R_1) if and only if $(\bigoplus_{\ell_1(n)} X, \bigoplus_{\ell_1(n)} Y, \mathcal{H})$ exhibits property-(R_1).*

For a Banach space X, recall the notations defined before Section 1.2. For $F \in \mathcal{F}(X)$, we identify $F \times F \times \ldots \times F(n-\text{times})$ with $\{(x_1, x_2, \ldots x_n, 0, 0 \ldots) : x_i \in F\}$. Let $\mathfrak{F} = \{\Pi_{i=1}^n F_i : F_i = F, F \in \mathcal{F}(X), n \in \mathbb{N}\}$. $\mathfrak{F}$ is now identified with a subfamily of $\mathcal{B}(X_1)$, more precisely a subfamily of $\mathcal{F}(X_1)$.

Theorem 5.5. *Let Y be a subspace of X. Then, $(X, Y, \mathcal{F}(X))$ exhibits property-(R_1) if and only if $(X_1, Y_1, \mathfrak{F})$ exhibits property-(R_1).*

Proof. Here, proving that the condition is sufficient concludes the proof.

Let $\mathcal{W} \in \mathfrak{F}$, $y \in Y_1$, and $r_1, r_2 > 0$ be such that $r(y, \mathcal{W}) < r_1 + r_2$ and $S_{r_2}(\mathcal{W}) \cap Y_1 \neq \emptyset$. Then, clearly $\mathcal{W} = \Pi_{i=1}^N F_i$ and we obtain a large $l \in \mathbb{N}(l > N)$ such that $\|y_i\| < r_1 + r_2$, for all $i \geq l$. Let $\mathcal{W}_l = \{(x_1, x_2, \ldots, x_l) : \exists\ (w_i) \in \mathcal{W}, x_i = w_i, 1 \leq i \leq l\}$ and $\Lambda = (y_1, y_2, \ldots, y_l)$.

It is clear that $r(\Lambda, \mathcal{W}_l) < r_1 + r_2$ and $S_{r_2}(\mathcal{W}_l) \cap \bigoplus_{\ell_1(l)} Y \neq \emptyset$.

Now, from Theorem 5.4, $S_{r_2}(\mathcal{W}_l) \cap B[\Lambda, r_1] \cap \bigoplus_{\ell_1(l)} Y \neq \emptyset$ is obtained.

Let $(z_1, \cdots, z_l)$ be in the intersection above. Let $z = (z_1, \cdots, z_l, 0, \ldots)$. Then, $z \in S_{r_2}(\mathcal{W}) \cap B[y, r_1] \cap Y_1$. Hence, the conclusion follows. □

However, it is not known whether the ℓ_1-sum remains stable for $(X_1, Y_1, \mathcal{F}(X_1))$.

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