ON ABSOLUTELY NORM ATTAINING OPERATORS

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Abstract. We give necessary and sufficient conditions for a bounded operator defined between complex Hilbert spaces to be absolutely norm attaining. We discuss structure of such operators in the case of self-adjoint and normal operators separately. Finally, we discuss several properties of absolutely norm attaining operators.

1. Introduction and Preliminaries

The class of absolutely norm attaining operators (shortly, $\mathcal{AN}$-operators) between complex Hilbert spaces were introduced and discussed several important class of examples and properties of these operators by Carvajal and Neves in [3]. Later, a structure of these operators on separable Hilbert spaces is proposed in [6]. But, an example of $\mathcal{AN}$-operator which does not fit into the characterization of [6] is given in [10] and the authors discussed the structure of positive $\mathcal{AN}$- operators between arbitrary Hilbert spaces. In this article, first, we give necessary and sufficient conditions for an operator to be *positive and* $\mathcal{AN}$. In fact, we show that a bounded operator T defined on an infinite dimensional Hilbert space is positive and $\mathcal{AN}$ if and only if there exists a unique triple (K, F, α), where K is a positive compact operator, F is a positive finite rank operator, α is positive real number such that $T = K-F+\alpha I$ and $KF = 0,\ F \leq \alpha I$ (See Theorem 2.4). In fact, here $\alpha = m_e(T)$, the essential minimum modulus of T. This is an improvement of [10, Theorem 5.1]. Using this result, we give explicit structure of self-adjoint and $\mathcal{AN}$-operators as well as normal and $\mathcal{AN}$-operators. Finally, we also obtain structure of general $\mathcal{AN}$-operators. In the process we also prove several important properties of $\mathcal{AN}$-operators. All these results are new.

We organize the article as follows: In the remaining part of this section we explain the basic terminology, notations and necessary results that will be needed for proving main theorems. In the second section

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we give a characterization of positive $\mathcal{AN}$-operators and prove several important properties. In the third section we discuss the structure of self-adjoint and normal $\mathcal{AN}$-operators and in the fourth section we discuss about general $\mathcal{AN}$-operators.

Throughout the article we consider complex Hilbert spaces which will be denoted by H, H_1, H_2 etc. The inner product and the induced norm are denoted by $\langle , \rangle$ and $\|\cdot\|$ respectively. The unit sphere of a closed subspace M of H is denoted by $S_M := \{x \in M : \|x\| = 1\}$ and P_M denote the orthogonal projection $P_M : H \to H$ with range M. The identity operator on M is denoted by I_M.

A linear operator $T : H_1 \to H_2$ is said to be bounded if there exists a $k > 0$ such that $\|Tx\| \leq k\|x\|$ for all $x \in H_1$. If T is bounded, the quantity $\|T\| = \sup\{\|Tx\| : x \in S_{H_1}\}$ is finite and is called the norm of T. We denote the space of all bounded linear operators between H_1 and H_2 by $\mathcal{B}(H_1, H_2)$. In case if $H_1 = H_2 = H$, then $\mathcal{B}(H_1, H_2)$ is denoted by $\mathcal{B}(H)$. For $T \in \mathcal{B}(H_1, H_2)$, there exists a unique operator denoted by $T^* : H_2 \to H_1$ satisfying

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x \in H_1 \text{ and for all } y \in H_2.$$

This operator T^* is called the adjoint of T. The null space and the range spaces of T are denoted by $N(T)$ and $R(T)$ respectively.

Let $T \in \mathcal{B}(H)$. Then T is said to be *normal* if $T^*T = TT^*$, *self-adjoint* if $T = T^*$. If T is self-adjoint and $\langle Tx, x\rangle \geq 0$ for all $x \in H$, then T is called *positive*. It is well known that for a positive operator T, there exists a unique positive operator $S \in \mathcal{B}(H)$ such that $S^2 = T$. We write $S = T^{\frac{1}{2}}$ and is called as the *positive square root* of T.

If $S, T \in \mathcal{B}(H)$ are self-adjoint and $S - T \geq 0$, then we write this by $S \geq T$.

If $P \in \mathcal{B}(H)$ is such that $P^2 = P$, then P is called a *projection*. If $N(P)$ and $R(P)$ are orthogonal to each other, then P is called an *orthogonal projection*. It is a well known fact that a projection P is an orthogonal projection if and only if it is self-adjoint if and only if it is normal.

We call an operator $V \in \mathcal{B}(H_1, H_2)$ to be an *isometry* if $\|Vx\| = \|x\|$ for each $x \in H_1$. An operator $V \in \mathcal{B}(H_1, H_2)$ is said to be a *partial isometry* if $V|_{N(V)^\perp}$ is an isometry. That is $\|Vx\| = \|x\|$ for all $x \in N(V)^\perp$. If $V \in \mathcal{B}(H)$ is isometry and onto, then V is said to be a *unitary operator*.

In general, if $T \in \mathcal{B}(H_1, H_2)$, then $T^*T \in \mathcal{B}(H_1)$ is positive and $|T| := (T^*T)^{\frac{1}{2}}$ is called the *modulus* of T. In fact, there exists a unique partial isometry $V \in \mathcal{B}(H_1, H_2)$ such that $T = V|T|$ and $N(V) = N(T)$. This factorization is called the *polar decomposition* of T.

If $T \in \mathcal{B}(H)$, then $T = \frac{T+T^*}{2} + i(\frac{T-T^*}{2i})$. The operators $Re(T) := \frac{T+T^*}{2}$ and $Im(T) := \frac{T-T^*}{2i}$ are self-adjoint and called the *real* and the *imaginary* parts of T respectively.

A closed subspace M of H is said to be *invariant* under $T \in \mathcal{B}(H)$ if $TM \subseteq M$ and *reducing* if both M and $M^\perp$ are invariant under T.

For $T \in \mathcal{B}(H)$, the set

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I : H \to H \text{ is invertible and } (T - \lambda I)^{-1} \in \mathcal{B}(H)\}$$

is called the resolvent set and the complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. It is well known that $\sigma(T)$ is a non empty compact subset of $\mathbb{C}$. The point spectrum of T is defined by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one-to-one}\}.$$

Note that $\sigma_p(T) \subseteq \sigma(T)$.

A self-adjoint operator $T \in \mathcal{B}(H)$ is positive if and only if $\sigma(T) \subseteq [0, \infty)$.

If $T \in \mathcal{B}(H_1, H_2)$, then T is said to be *compact* if for every bounded set S of H_1, the set $T(S)$ is pre-compact in H_2. Equivalently, for every bounded sequence (x_n) of H_1, (Tx_n) has a convergent subsequence in H_2. We denote the set of all compact operators between H_1 and H_2 by $\mathcal{K}(H_1, H_2)$. In case if $H_1 = H_2 = H$, then $\mathcal{K}(H_1, H_2)$ is denoted by $\mathcal{K}(H)$.

A bounded linear operator $T : H_1 \to H_2$ is called *finite rank* if $R(T)$ is finite dimensional. The space of all finite rank operators between H_1 and H_2 is denoted by $\mathcal{F}(H_1, H_2)$ and we write $\mathcal{F}(H, H) = \mathcal{F}(H)$.

All the above mentioned basics of operator theory can be found in [13, 4, 2, 12].

An operator $T \in \mathcal{B}(H_1, H_2)$ is said to be *norm attaining* if there exists a $x \in S_{H_1}$ such that $\|Tx\| = \|T\|$. We denote the class of norm attaining operators by $\mathcal{N}(H_1, H_2)$. It is known that $\mathcal{N}(H_1, H_2)$ is dense in $\mathcal{B}(H_1, H_2)$ with respect to the operator norm of $\mathcal{B}(H_1, H_2)$. We refer [5] for more details on this topic.

We say $T \in \mathcal{B}(H_1, H_2)$ to be *absolutely norm attaining* or $\mathcal{AN}$-operator (shortly), if $T|_M$, the restriction of T to M, is norm attaining for every non zero closed subspace M of H_1. That is $T|_M \in \mathcal{N}(M, H_2)$ for every non zero closed subspace M of H_1 [3]. This class contains $\mathcal{K}(H_1, H_2)$, and the class of partial isometries with finite dimensional null space or finite dimensional range space.

We have the following characterization of norm attaining operators:

Proposition 1.1. [3, Proposition 2.4] *Let $T \in \mathcal{B}(H)$ be self-adjoint. Then*

(1) *$T \in \mathcal{N}(H)$ if and only if either $\|T\| \in \sigma_p(T)$ or $-\|T\| \in \sigma_p(T)$*
(2) *if $T \geq 0$, then $T \in \mathcal{N}(H)$ if and only if $\|T\| \in \sigma_p(T)$.*

For $T \in \mathcal{B}(H_1, H_2)$, the quantity

$$m(T) := \inf \{\|Tx\| : x \in S_{H_1}\}$$

is called the minimum modulus of T. If $H_1 = H_2 = H$ and $T^{-1} \in \mathcal{B}(H)$, then $m(T) = \dfrac{1}{\|T^{-1}\|}$ (see [1, Theorem 1] for details).

The following definition is available in [9] for densely defined closed operators (not necessarily bounded) on a Hilbert space, and this holds true automatically for bounded operators.

Definition 1.2. [9, Definition 8.3 page 178] Let $T = T^* \in \mathcal{B}(H)$. Then the *discrete spectrum* $\sigma_d(T)$ of T is defined as the set of all eigenvalues of T with finite multiplicities which are isolated points of the spectrum $\sigma(T)$ of T. The complement set $\sigma_{ess}(T) = \sigma(T) \setminus \sigma_d(T)$ is called the *essential spectrum* of T.

By the Weyl's theorem we can assert that if $T = T^*$ and $K = K^* \in \mathcal{K}(H)$, then $\sigma_{ess}(T + K) = \sigma_{ess}(T)$ (see [9, Corollary 8.16, page 182] for details). If H is a separable Hilbert space, the *essential minimum modulus* of T is defined to be $m_e(T) := \inf \{\lambda : \lambda \in \sigma_{ess}(|T|)\}$ (see [1] for details). The same result in the general case is dealt in [8, Proposition 2.1].

Let $H = H_1 \oplus H_2$ and $T \in \mathcal{B}(H)$. Let $P_j : H \to H$ be an orthogonal projection onto H_j for $j = 1, 2$. Then $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, where $T_{ij} : H_j \to H_i$ is the operator given by $T_{ij} = P_i T P_j|_{H_j}$. In particular, $T(H_1) \subseteq H_1$ if and only if $T_{12} = 0$. Also, H_1 reduces T if and only if $T_{12} = 0 = T_{21}$ (for details see [12, 4].

2. Positive $\mathcal{AN}$-operators

In this section we describe the structure of operators which are positive and satisfy the $\mathcal{AN}$-property. First, we recall results which are necessary for proving our results.

Theorem 2.1. [10, Theorem 5.1] *Let H be a complex Hilbert space of arbitrary dimension and let P be a positive operator on H. Then P is an $\mathcal{AN}$- operator iff P is of the form $P = \alpha I + K + F$, where $\alpha \geq 0$, K is a positive compact operator and F is self-adjoint finite rank operator.*

Theorem 2.2. [10, Theorem 3.8] *Let $T \in B(H)$ be positive and $T \in \mathcal{AN}(H)$. Then*

$$T = \sum_{\alpha \in \Lambda} \beta_\alpha v_\alpha \otimes v_\alpha, \tag{2.1}$$

where $\{v_\alpha : \alpha \in \Lambda\}$ is an orthonormal basis consisting of entirely eigenvectors of T and for every $\alpha \in \Lambda$, $Tv_\alpha = \beta_\alpha v_\alpha$ with $\beta_\alpha \geq 0$ such that

(1) *for every non empty set Γ of Λ, we have*

$$\sup\{\beta_\alpha : \alpha \in \Gamma\} = \max\{\beta_\alpha : \alpha \in \Gamma\}$$

(2) *the spectrum $\sigma(T) = \overline{\{\beta_\alpha : \alpha \in \Lambda\}}$ has at most one limit point. Moreover, this unique limit point (if exists) can only be the limit of an increasing sequence in the spectrum*
(3) *the set $\{\beta_\alpha : \alpha \in \Lambda\}$ of eigenvalues of T, without counting multiplicities, is countable and has atmost one eigenvalue with infinite multiplicity*
(4) *if $\sigma(T)$ has both, a limit point and an eigenvalue with infinite multiplicity, then they must be same.*

(Here $(v_\alpha \otimes v_\alpha)(x) = \langle x, v_\alpha\rangle v_\alpha$ for each $\alpha \in \Lambda$ and for each $x \in H$).

Lemma 2.3. *Let $S, T \in \mathcal{B}(H)$ be positive such that $S \leq T$. Then $N(T) \subseteq N(S)$.*

Proof. If $x \in H$, then $\|S^{\frac{1}{2}}x\|^2 = \langle Sx, x\rangle \leq \langle Tx, x\rangle = \|T^{\frac{1}{2}}x\|^2$. By observing the fact that for any $A \geq 0$, $N(A^{\frac{1}{2}}) = N(A)$, the conclusion follows. □

Theorem 2.4. *Let H be an infinite dimensional Hilbert space and $T \in \mathcal{B}(H)$. Then the following statements are equivalent:*

(1) *$T \in \mathcal{AN}(H)$ and positive*
(2) *there exists a unique triple (K, F, α) where*
   (a) *$K \in \mathcal{K}(H)$ is positive*
   (b) *$F \in \mathcal{F}(H)$ and $0 \leq F \leq \alpha I$*
   (c) *$KF = 0$*

   such that $T = K - F + \alpha I$.

Proof. Proof of (1) ⇒ (2): By Theorem 2.1, $T = K' - F' + \alpha I$, where $K' \in \mathcal{K}(H)$ is positive, $F' = F'^* \in \mathcal{F}(H)$ and $\alpha \geq 0$. Next we claim that $K'F' = 0$. This readily follows by the proof in [10, Theorem 5.1].

Now, let $F' = F'_+ - F'_-$ be the decomposition of F' in terms of positive operators F'^+ and F'_-, respectively (see [7, page 180] for details). Note that $F'_+F'_- = 0$. Write $K = K' + F'_-$ and $F = F'_+$. Then $K \geq 0$ and $F \geq 0$. Since $K'F' = 0$, it follows that $K'|F'| = 0$. That is $K'(F'_+ + F'_-) = 0$. Also, $K'(F'_+ - F'_-) = 0$. These two equations imply that $KF = 0$. As $T \geq 0$ and $F \geq 0$ such that $TF = FT$, it follows that $FT \geq 0$. But $FT = F(\alpha I - F)$. Let $\lambda \in \sigma(F)$. Then $\lambda \geq 0$ and since $FT \geq 0$, by the spectral mapping theorem, we have that $\lambda(\alpha - \lambda) \geq 0$. From this, we can conclude that $\alpha - \lambda \geq 0$ for each $\lambda \in \sigma(F)$. As $\alpha I - F$ is self-adjoint and $\sigma(\alpha I - F) \subseteq [0, \infty)$, $\alpha I - F$ must be positive. This concludes that $F \leq \alpha I$.

Next we show that the triple satisfying the given conditions is unique. Suppose there exists two triples $(K_1, F_1, \alpha_1), (K_2, F_2, \alpha_2)$ satisfying the stated conditions. We prove this by considering all possible cases.

Case 1; $\alpha_1 = 0$: In this case, $F_1 = 0$. Hence $K_1 = T = K_2 - F_2 + \alpha_2 I$. This shows that $\alpha_2 I = K_1 - K_2 + F_2$, a compact operator. Since H is infinite dimensional, it follows that $\alpha_2 = 0$. Thus $F_2 = 0$. Hence we can conclude that $K_1 = K_2$.

Case 2; $F_1 = 0, \alpha_1 > 0$: In this case,

$$K_1 + \alpha_1 I = K_2 - F_2 + \alpha_2 I. \tag{2.2}$$

Then $(\alpha_2 - \alpha_1)I = (K_1 - K_2) + F_2$, a compact operator. If this is zero, then $\alpha_1 = \alpha_2$. If not, $(\alpha_1 - \alpha_2)I$ is a compact operator and H is infinite dimensional, $\alpha_1 = \alpha_2$.

Now, the Equation (2.2) can be written as $K_2 = F_2 + K_1 \geq F_2$. Now, by Lemma 2.3, we have that $N(K_2) \subseteq N(F_2)$. But, by the condition $K_2 F_2 = 0$, we have, $R(F_2) \subseteq N(K_2)$, hence $R(F_2) \subseteq N(F_2)$. Thus, $F_2 = 0$. From this we can conclude that $K_1 = K_2$.

Case 3 $K_1 = 0,\ F_1 \neq 0,\ \alpha_1 > 0$: We have $F_1 + \alpha_1 I = K_2 - F_2 + \alpha_2 I$. Using the same argument as in the above cases, we can conclude that $\alpha_1 = \alpha_2$. Thus we have $F_2 = K_2 + F_1 \geq K_2$. Now, by Lemma 2.3, $N(F_2) \subseteq N(K_2)$. But by the property $K_2 F_2 = 0$, it follows that $R(F_2) \subseteq N(K_2)$. Hence $H = N(F_2) \oplus R(F_2) \subseteq N(K_2)$. This shows that $K_2 = 0$. Finally, using this we can get $F_1 = F_2$.

Case 4 $K_1 \neq 0,\ F_1 \neq 0,\ \alpha_1 > 0$: We can prove $\alpha_1 = \alpha_2$ by arguing as in the earlier cases. With this we have

$$K_1 - F_1 = K_2 - F_2. \tag{2.3}$$

As F_1 commute with K_1 and F_1, it commute with $K_2 - F_2$. So F_1 must commute with $(K_2 - F_2)^2 = K_2^2 + F_2^2 = (K_2 + F_2)^2$. Thus, it commute with $K_2 + F_2$. Hence we can conclude that F_1 commute with both K_2 and F_2. Since $N(F_1)$ is invariant under K_1 and F_1, by Equation (2.1), $N(F_1)$ is invariant under $K_2 - F_2$.

Now if $x \in N(F_1)$. Then by Equation (2.3), we have $(K_2 - K_1)x = F_2 x$. Using the fact that $F_2 \geq 0$, we can conclude that $K_2 \geq K_1$ on $N(F_1)$. We also show that this will happen on $R(F_1)$.

For $x \in H$, we have $F_1x \in R(F_1)$. Now,

$$\langle (F_2 - F_1)(F_1x), F_1x\rangle = \langle (K_2 - K_1)(F_1x), F_1x\rangle = \langle K_2(F_1x), F_1x\rangle \geq 0.$$

This shows that $K_2 - K_1 = F_2 - F_1 \geq 0$ on $R(F_1)$. Combining with the earlier argument, we can conclude that $K_1 \leq K_2$. Now, interchanging the roles of K_1 and K_2, we can conclude that $K_2 \leq K_1$ and hence $K_1 = K_2$. By Equation (2.3), we can conclude that $F_1 = F_2$.

Proof of (2) $\Rightarrow$ (1): If $T = K - F + \alpha I$, where $K \in \mathcal{K}(H)$ is positive, $F \in \mathcal{F}(H)$ is positive, $\alpha \geq 0$ and $KF = 0$. Then by Theorem 2.1, $T \in \mathcal{AN}(H)$. Since $K \geq 0$ and $-F + \alpha I \geq 0$, T must be positive. $\square$

Remark 2.5. Let T be as in Theorem 2.4. Then we have the following:

(1) if $\alpha = 0$, then $F = 0$ and hence $T = K$. In this case $\sigma_{ess}(T) = \{\alpha\}$
(2) if $\alpha > 0$ and $F = 0$, then $T = K + \alpha I$. In this case, $\sigma_{ess}(T) = \{\alpha\}$ and $m_e(T) = \alpha = m(T)$
(3) if $\alpha > 0$, $K = 0$ and $F \neq 0$, then $T = \alpha I - F$. In this case also, $\sigma_{ess}(T) = \{\alpha\}$ and $m_e(T) = \alpha$
(4) if $\alpha > 0$, $F \neq 0$ and $K \neq 0$, then by the Weyl's theorem, $\sigma_{ess}(T) = \{\alpha\}$ and $m_e(T) = \alpha$
(5) if $\alpha = 0$ and $K = 0$, then $T = 0$
(6) if $N(T)$ is infinite dimensional, then 0 is an eigenvalue with infinite multiplicity and hence $\alpha = 0$ by Theorem 2.2. In this case, $F = 0$ and hence $T = K$.

Remark 2.6. If we take $F = 0$ in Theorem 2.4, then we get the structure obtained in [6].

Here we prove some important properties of $\mathcal{AN}$-operators.

Proposition 2.7. *Let $T = K - F + \alpha I$, where $K \in \mathcal{K}(H)$ is positive, $F \in \mathcal{F}(H)$ is positive with $KF = 0$ and $F \leq \alpha I$. Assume that $\alpha > 0$. Then the following statements hold.*

(1) *$R(T)$ is closed*
(2) *$N(T)$ is finite dimensional*
(3) *$N(T) \subseteq N(K)$*
(4) *$Fx = \alpha x$ for all $x \in N(T)$. Hence $N(T) \subseteq R(F)$. In this case, $\|F\| = \alpha$.*
(5) *T is one-to-one if and only if $\|F\| < \alpha$*
(6) *T is Fredholm and $m_e(T) = \alpha$.*

Proof. Proof of (1): Since $K - F$ is a compact operator, $R(T)$ is closed. Here we have used the fact that for any $A \in \mathcal{K}(H)$, and $\lambda \in \mathbb{C} \setminus \{0\}$, $R(K + \lambda I)$ is closed.

Proof of (2): Let $x \in N(T)$. Then

$$(K - F)x = -\alpha x. \tag{2.4}$$

That is $\alpha I_{N(T)}$ is compact. This concludes that $N(T)$ is finite dimensional.

Proof of (3): Let $x \in N(T)$. Multiplying Equation (2.4) by K and using the fact that $KF = FK = 0$, we have $K^2x = -\alpha Kx$. If $Kx \neq 0$, then $-\alpha \in \sigma_p(K)$, contradicts the positivity of K. Hence $Kx = 0$.

Proof of (4): Clearly, if $Tx = 0$, then by (3), we have $Fx = \alpha x$. This also concludes that $N(T) \subseteq R(F)$.

Proof of (5): If T is not one-to-one, then $Fx = \alpha x$ for $x \in N(T)$ by (4). Suppose T is one-to-one and $\|F\| = \alpha$. Since F is norm attaining by Proposition 1.1, there exists $x \in S_H$ such that $Fx = \alpha x$. Then $Tx = Kx - Fx + \alpha x = Kx$. But $KF = 0$ implies that $x \in N(K)$. So, $Tx = Kx$=0. By the injectivity of T, we have that $x = 0$. This contradicts the fact that $x \in S_H$. Hence $\|F\| < \alpha$.

Proof of (6): Note that $\sigma_{ess}(T) = \{\alpha\}$ by the Weyl's theorem on essential spectrum. Hence $m_e(T) = \alpha = m_e(T^*)$. Now T is Fredholm operator by [1, Theorem 2] with index zero. $\square$

Theorem 2.8. *Let $T \in \mathcal{B}(H)$ and positive. Then $T \in \mathcal{AN}(H)$ if and only if $T^2 \in \mathcal{AN}(H)$.*

Proof. First we will assume that $T \in \mathcal{AN}(H)$. Then there exists a triple (K, F, α) as in (2) of Theorem 2.4. Then $T^2 = K_1 - F_1 + \beta I$, where $K_1 = K^2 + 2\alpha K$, a positive compact operator, $F_1 = 2\alpha F - F^2 = (2\alpha I - F)F$ and $\beta = \alpha^2$. Clearly, $F_1 \geq 0$ as it is the product of two commuting positive operators. Also $F_1 \in \mathcal{F}(H)$. Next, we show that $F_1 \leq \alpha^2 I$. Clearly, $\alpha^2 I - F_1$ is self-adjoint and $\alpha^2 I - F_1 = (\alpha I - F)^2 \geq 0$. It can be easily verified that $K_1 F_1 = 0$. So, T^2 is also in the same form. Hence by Theorem 2.4, $T^2 \in \mathcal{AN}(H)$.

Now, let $T^2 \in \mathcal{AN}(H)$. Then by Theorem 2.4, $T^2 = K - F + \alpha I$, where $K \in \mathcal{K}(H)$ is positive, $F \in \mathcal{F}(H)$ is positive with $FK = KF = 0$ and $F \leq \alpha I$. If $\alpha > 0$, then $(T - \sqrt{\alpha} I)(T + \sqrt{\alpha} I) = K - F$. Since T is positive $T + \sqrt{\alpha} I$ is a positive invertible operator. Hence $T - \sqrt{\alpha} I = (K - F)(T + \sqrt{\alpha} I)^{-1}$. Hence there is a positive compact operator, namely $K_1 = K(T + \sqrt{\alpha})^{-1}$ and a finite rank positive operator, namely $F_1 = F(T + \sqrt{\alpha} I)^{-1}$, such that $T - \sqrt{\alpha} I = K_1 + F_1$. Hence $T = K_1 - F_1 + \sqrt{\alpha} I$. Also note that since F and K commute with T^2, hence with T. Thus, we can conclude that $F_1K_1 = 0$. Finally,

$$\begin{aligned}\|F_1\| \le \|F\|\,\|(T+\sqrt{\alpha}I)^{-1}\| &\le \alpha\,\frac{1}{m(T+\sqrt{\alpha}I)}\\ &= \frac{\alpha}{\sqrt{\alpha}+m(T)}\\ &\le \frac{\alpha}{\sqrt{\alpha}} = \sqrt{\alpha}.\end{aligned}$$

In the third step of the above inequalities we used the fact that $m(T+\sqrt{\alpha}I) = \sqrt{\alpha}+m(T)$, which follows by [6, Proposition 2.1].

If $\alpha = 0$, then clearly $F = 0$ and hence $T^2 = K$. So, $T = K^{\frac{1}{2}}$, a compact operator which is clearly an $\mathcal{AN}$-operator. □

Corollary 2.9. *Let $T \in \mathcal{B}(H)$ and positive. Then $T \in \mathcal{AN}(H)$ if and only if $T^{\frac{1}{2}} \in \mathcal{AN}(H)$.*

Proof. Let $S = T^{\frac{1}{2}}$. Then $S \ge 0$. The conclusion follows by Theorem 2.8. □

Corollary 2.10. *Let $T \in \mathcal{B}(H_1, H_2)$. Then $T \in \mathcal{AN}(H_1, H_2)$ if and only $T^*T \in \mathcal{AN}(H_1)$.*

Proof. Proof follows from the following: $T^*T \in \mathcal{AN}(H_1) \Leftrightarrow |T|^2 \in \mathcal{AN}(H_1) \Leftrightarrow |T| \in \mathcal{AN}(H_1) \Leftrightarrow T \in \mathcal{AN}(H_1, H_2)$. □

We have the following consequence.

Theorem 2.11. *Let $T \in \mathcal{AN}(H)$ be self-adjoint and λ be a purely imaginary number. Then $T \pm \lambda I \in \mathcal{AN}(H)$.*

Proof. Let $S = T \pm \lambda I$. Then $S^*S = T^2 + |\lambda|^2 I = K - F + (\alpha + |\lambda|^2)I$, where the triple (K, F, α) satisfy conditions (2) of Theorem 2.4. Hence by Corollary 2.10, $S \in \mathcal{AN}(H)$. □

The following result is well known.

Lemma 2.12. *Let $S, T \in \mathcal{B}(H)$ be such that $S^{-1}, T^{-1} \in \mathcal{B}(H)$. Then $S^{-1} - T^{-1} = T^{-1}(T - S)S^{-1}$.*

Theorem 2.13. *Let $T = K - F + \alpha I$, where (K, F, α) satisfy conditions (2) of Theorem 2.4. Then*

(1) *$R(F)$ reduces T*

(2) $T = \begin{pmatrix} K_0 + \alpha I|_{N(F)} & 0 \\ & \alpha I|_{R(F)} - F_0 \end{pmatrix}$, *where $K_0 = K|_{N(F)}$ and $F_0 = F|_{R(F)}$.*

(3) *if T is one-to-one and $\alpha > 0$, then $T^{-1} \in \mathcal{B}(H)$ and*

$$T^{-1} = \begin{pmatrix} \alpha^{-1}I_{N(F)} - \alpha^{-1}K_0(K_0+\alpha I_{N(F)})^{-1} & 0 \\ 0 & \alpha^{-1}I_{R(F)} + \alpha^{-1}F_0(\alpha I_{R(F)} - F_0)^{-1} \end{pmatrix}.$$

Proof. Proof of (1): First note that $T \geq 0$ and $T \in \mathcal{AN}(H)$. Let $y = Fx$ for some $x \in H$. Then $Ty = TFx = (K - F + \alpha I)Fx = (\alpha I - F)(Fx) = F(\alpha I - F)x \in R(F)$. This shows that $R(F)$ is invariant under T. As T is positive, it follows that $R(F)$ is a reducing subspace for T.

Proof of (2): First, we show that K_0 is a map on $N(F)$. For this we show that $N(F)$ invariant under K. If $x \in N(F)$, then $FKx = 0$ since $FK = 0$. This proves that $N(F)$ is invariant under K. Thus $K_0 \in \mathcal{K}(N(F))$. Also, clearly, $R(F)$ is invariant under F. Thus $F_0 : R(F) \to R(F)$ is a finite dimensional operator. With respect to the pair of subspaces $(N(F), R(F))$, K has the decomposition:

$$\begin{pmatrix} K_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly the operators F and αI has the following block matrix forms respectively:

$$\begin{pmatrix} 0 & 0 \\ 0 & F_0 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha I_{N(F)} & 0 \\ 0 & \alpha I_{R(F)} \end{pmatrix}.$$

With these representation of K, F and αI, by definition, T can be represented as in (2).

Proof of (3): By (1) of Proposition 2.7, $R(T)$ is closed. As T is one-to-one, T is bounded below. Since T is positive, $T^{-1} \in \mathcal{B}(H)$. In this case $\|F_0\| = \|F\| < \alpha$, by (5) of Proposition 2.7. Hence we have

$$T^{-1} = \begin{pmatrix} (K_0 + \alpha I_{N(F)})^{-1} & 0 \\ 0 & (\alpha I_{R(F)} - F_0)^{-1} \end{pmatrix}. \tag{2.5}$$

By Lemma 2.12, we have

$$(K_0 + \alpha I_{N(F)})^{-1} - \alpha^{-1}I_{N(F)} = \alpha^{-1}I_{N(F)} - \alpha^{-1}K_0(K_0 + \alpha I_{N(F)})^{-1},$$

and hence

$$(K_0 + \alpha I_{N(F)})^{-1} = \alpha^{-1}I_{N(F)} - \alpha^{-1}I_{N(F)} - \alpha^{-1}K_0(K_0 + \alpha I_{N(F)})^{-1}.$$

Substituting these quantities in Equation 2.5, we obtain the representation of T^{-1} as in (3). $\square$

Remark 2.14. Let

$$\beta = \alpha^{-1},$$

$$K_1 = \begin{pmatrix} \alpha^{-1}K_0(K_0 + \alpha I_{N(F)})^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$F_1 = \begin{pmatrix} 0 & 0 \\ 0 & \alpha^{-1}F_0(\alpha I_{R(F)} - F_0)^{-1} \end{pmatrix}.$$

Then $T^{-1} = \beta I - K_1 + F_1$. Note that $\|K_1\| \leq \beta$, since $\|K_0(\alpha I_{N(F)} + K_0)^{-1}\| \leq 1$. Clearly, by definition, $K_1F_1 = 0$. This is exactly, the structure of absolutely minimum attaining operators (shortly $\mathcal{AM}$-operators) in case when T is positive and one-to-one. We refer [11] for more details of the structure of these operators. We recall that $A \in \mathcal{B}(H_1, H_2)$ is said to be minimum attaining if there exists $x_0 \in S_{H_1}$ such that $\|Ax_0\| = m(A)$ and absolutely minimum attaining if $A|_M$ is minimum attaining for each non zero closed subspace M of H_1.

Proposition 2.15. *Let $T \in \mathcal{B}(H)$ be satisfying conditions in Theorem 2.4. Then with respect the pair of subspace $(N(K), N(K)^\perp)$, T has the following decomposition:*

$$T = \begin{pmatrix} \alpha I_{N(K)} - F_0 & 0 \\ 0 & K_0 + \alpha I_{N(K)^\perp} \end{pmatrix},$$

where $F_0 = F|_{N(K)}$ and $K_0 = K|_{N(K)^\perp}$.

Proof. First we show that $N(K)$ is a reducing subspace for T. We know by Theorem 2.4, that T is positive. Hence it suffices to show that $N(K)$ is invariant under T. For this, let $x \in N(K)$. Then $Tx = (\alpha I - F)(x)$ and $K(Tx) = (\alpha I - F)(Kx) = 0$. This proves the claim. Next, if $x \in N(K)$, then $Tx = (\alpha I - F)(x)$. That is $T|_{N(K)} = I_{N(K)} - F|_{N(K)}$.

If $y \in N(K)^\perp = \overline{R(K)}$, then there exists a sequence $(x_n) \subset H$ such that $y = \lim\limits_{n\to\infty} Kx_n$. So $Fy = \lim\limits_{n\to\infty} FKx_n = 0$. Thus we have $Ty = Ky + \alpha y$. So $T|_{N(K)^\perp} = K_{N(K)^\perp} + \alpha I_{N(K)^\perp}$. □

3. Self-adjoint and Normal $\mathcal{AN}$-operators

In this section, first we discuss the structure of self-adjoint $\mathcal{AN}$-operators. Later, we extend this to the case of normal operators.

Theorem 3.1. *Let $T = T^* \in \mathcal{AN}(H)$. Then there exists an orthonormal basis consisting of eigenvectors of T.*

Proof. The proof follows in the similar lines of [10, Theorem 3.1]. For the sake of completeness we provide the details here. Let $\mathcal{B} = \{x_\alpha : \alpha \in I\}$ be the maximal set of orthonormal eigenvectors of T. This set is non empty, as $T = T^* \in \mathcal{AN}(H)$. Let $M = \overline{\text{span}}\{x_\alpha : \alpha \in I\}$. Then we claim that $M = H$. If not, $M^\perp$ is a proper non-zero closed subspace of H and it is invariant under T. Since $T = T^* \in \mathcal{AN}(H)$, then we have either $||T|M^{\perp}||$ or $-||T|M^{\perp}||$ is an eigenvalue for $T|M^{\perp}$. Hence there is a non-zero vector, say x_0 in $M^{\perp}$, such that $Tx_0 = \pm||T|M^{\perp}||x_0$. Since $M \cap M^{\perp} = \{0\}$, we have arrived to a contradiction to the maximality of $\mathcal{B}$. □

Proposition 3.2. *Let $T = T^* \in \mathcal{AN}(H)$. Then the following holds:*

(1) *T can have atmost two eigenvalues with infinite multiplicity. Moreover, if α and β are such eigenvalues, then $\alpha = \pm\beta$*
(2) *if T has an eigenvalue α with infinite multiplicity and β is a limit point of $\sigma(T)$, then $\alpha = \pm\beta$*
(3) *$\sigma(T)$ can have atmost two limit points. If α and β are such points, then $\alpha = \pm\beta$.*

Proof. Proof of (1): Let $\alpha_j \in \sigma_p(T)$ be such that $N(T - \alpha_j I)$ is infinite dimensional for each $j = 1, 2, 3$. Then $\alpha_j^2 \in N(T^2)$ and we have $N(T - \alpha_j I) \subseteq N(T^2 - \alpha_j^2 I)$ for each $j = 1, 2, 3$. Since $T^2 \in \mathcal{AN}(H)$ and positive, by (3) of Theorem 2.2, it follows that $\alpha_1^2 = \alpha_2^2 = \alpha_3^2$. Thus $\alpha_1 = \pm\alpha_2 = \pm\alpha_3$.

Proof of (2): Let $\alpha \in \sigma_p(T)$ with infinite multiplicity and $\beta \in \sigma(T)$, which is a limit point. Since $\sigma(T^2) = \{\lambda^2 : \lambda \in \sigma(T)\}$, it follows that α^2 is an eigenvalue of T^2 with infinite multiplicity as $N(T - \alpha I) \subseteq N(T^2 - \alpha^2 I)$ and β^2 is a limit point $\sigma(T^2)$. Since $T^2 \in \mathcal{AN}(H)$ is positive, by (4) of Theorem (2.2), $\alpha^2 = \beta^2$. Thus $\alpha = \pm\beta$.

Proof of (3): Let $\alpha, \beta \in \sigma(T)$ be limit points of $\sigma(T)$. Then $\alpha^2, \beta^2 \in \sigma(T^2)$ are limit points of $\sigma(T^2)$ and since $T^2 \in \mathcal{AN}(H)$ and positive, by (2) of Theorem 2.2, $\alpha^2 = \beta^2$, concluding $\alpha = \pm\beta$. By arguing as in Proof of (1), we can show that there are at most two limit points for the spectrum. □

Let $T = T^* \in \mathcal{B}(H)$ and have the polar decomposition $T = V|T|$. Let $H_0 = N(T)$, $H_+ = N(I - V)$ and $H_- = N(I + V)$. Then $H = H_0 \oplus H_+ \oplus H_-$. All these subspaces are invariant under T. Let $T_0 = T|_{N(T)}$, $T_+ = T|_{H_+}$ and $T_- = T|_{H_-}$. Then $T = T_0 \oplus T_+ \oplus T_-$. Further more, $T+$ is strictly positive, T_- is strictly negative and $T_0 = 0$ if $N(T) \neq \{0\}$. Let $P_0 = P_{N(T)}$, $P_\pm = P_{H_\pm}$. Then $P_0 = I - V^2$ and $P_\pm = \frac{1}{2}(V^2 \pm V)$. Thus $V = P_+ - P_-$. For details see [9, Example 7.1, page 139]. Note that the operators T_+ and T_- are different than those used in Theorem 2.4.

Theorem 3.3. *Let $T \in \mathcal{AN}(H)$ be self-adjoint with the polar decomposition $T = V|T|$. Then*

(1) *the operator T has the representation:*

$$T = K - F + \alpha V,$$

where $K \in \mathcal{K}(H),\ F \in \mathcal{F}(H)$ *are self-adjoint with* $KF = 0$ *and* $F^2 \leq \alpha^2 I$

(2) *if* T *is not a compact operator, then* $V \in \mathcal{AN}(H)$

(3) $K^2 + 2\alpha Re(VK) \geq 0.$

Proof. Proof of (1): We prove this in two cases;
Case 1 : T one-to-one: In this case $H = H_+ \oplus H_-$ and $T = T_+ \oplus T_-$. Since $H_\pm$ reduces T, we have $T_\pm \in \mathcal{B}(H_\pm)$. As $T \in \mathcal{AN}(H)$, we have that $T_\pm \in \mathcal{AN}(H_\pm)$. Hence By Theorem 2.4, we have that $T_+ = K_+ - F_+ + \alpha I_{H_+}$ such that K_+ is positive compact operator, F_+ is finite rank positive operator with the property that $K_+F_+ = 0$ and $F_+ \leq \alpha I_{H_+}$. As T_+ is strictly positive, $\alpha > 0$.

Similarly, $T_- \in \mathcal{AN}(H_-)$ and strictly negative. Hence there exists a triple (K_-, F_-, β) such that $-T_- = K_- - F_- + \beta I_{H_-}$, where $K_- \in \mathcal{K}(H_-)$ is positive, $F_- \in \mathcal{F}(H_-)$ is positive with $K_-F_- = 0,\ F_- \leq \beta I_{H_-}$ and $\beta > 0$. Hence we can write $T_- = -K_- + F_- - \beta I_{H_-}$ and

$$T = \begin{pmatrix} T_+ & 0 \\ 0 & T_- \end{pmatrix} = \begin{pmatrix} K_+ - F_+ + \alpha I_{H_+} & 0 \\ 0 & -K_- + F_- - \beta I_{H_-} \end{pmatrix}$$
$$= \begin{pmatrix} K_+ & 0 \\ 0 & -K_- \end{pmatrix} - \begin{pmatrix} -F_+ & 0 \\ 0 & F_- \end{pmatrix} + \begin{pmatrix} \alpha I_{H_+} & 0 \\ 0 & \beta I_{H_-} \end{pmatrix}.$$

We also have that

$$|T| = \begin{pmatrix} T_+ & 0 \\ 0 & -T_- \end{pmatrix}$$
$$= \begin{pmatrix} K_+ & 0 \\ 0 & K_- \end{pmatrix} - \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix} + \begin{pmatrix} \alpha I_{H_+} & 0 \\ 0 & \beta I_{H_-} \end{pmatrix}.$$

Let $K_1 := \begin{pmatrix} K_+ & 0 \\ 0 & K_- \end{pmatrix}$ and $F_1 := \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix}$. Then

$$|T| = K_1 - F_1 - \begin{pmatrix} \alpha I_{H_+} & 0 \\ 0 & \beta I_{H_-} \end{pmatrix}.$$

Clearly, $K_1F_1 = 0$, both K_1 and F_1 are positive, $F \leq \max\{\alpha, \beta\}I$. By the uniqueness of the decomposition (see Theorem 2.4), if $|T| = K_2 - F_2 + \gamma I$, then we can conclude that $K_1 = K_2,\ F_1 = F_2$ and $\alpha = \beta = \gamma$. With this observation, we have that

$$\begin{pmatrix} \alpha I_{H_+} & 0 \\ 0 & \beta I_{H_-} \end{pmatrix} = \alpha(P_+ - P_-) = \alpha V.$$

Now taking $K := \begin{pmatrix} K_+ & 0 \\ 0 & -K_- \end{pmatrix}$, $F := \begin{pmatrix} -F_+ & 0 \\ 0 & F_- \end{pmatrix}$, we can write $T = K - F + \alpha V$. Here K is self-adjoint compact operator, F is a self-adjoint finite rank operator with $KF = 0$. Finally, it is easy to verify that $F^2 \leq \alpha^2 I$.

Next, we show that $V \in \mathcal{AN}(H)$. Since T^{-1} exists, T cannot be compact. It suffices to prove $V^2 \in \mathcal{AN}(H)$. We have $V^2 = P_+ + P_- = P_{R(T)} = I \in \mathcal{AN}(H)$.

Case 2; T need not be one-to-one: In this case $T_0 = 0$ and $T = T_0 \oplus T_+ \oplus T_-$. Since all the operators T_0, T_+ and T_- are $\mathcal{AN}$-operators, we have that

$$
\begin{aligned}
T &= \begin{pmatrix} T_+ & 0 & 0 \\ 0 & T_- & 0 \\ 0 & 0 & T_0 \end{pmatrix} \\
&= \begin{pmatrix} K_+ - F_+ + \alpha I_{H_+} & 0 & 0 \\ 0 & -K_- + F_- - \beta I_{H_-} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} K_+ & 0 & 0 \\ 0 & K_- & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} -F_+ & 0 & 0 \\ 0 & F_- & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha I_{H_-} & 0 & 0 \\ 0 & -\alpha I_{H_-} & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

(Following the same arguments as in Case (1), we can show that $\alpha = \beta$)

Let $K = \begin{pmatrix} K_+ & 0 & 0 \\ 0 & K_- & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} -F_+ & 0 & 0 \\ 0 & F_- & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Clearly, $V = \begin{pmatrix} I_{H_+} & 0 & 0 \\ 0 & -I_{H_-} & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $T = K - F + \alpha V$ and K and F satisfy the stated properties.

Proof of (2): Note that if $\alpha = 0$, then T is compact. If $\alpha > 0$ and V is a finite rank operator, then also T can be compact. Hence assume that $\alpha > 0$ and $R(V)$ is infinite dimensional. But by Theorem (2.7), $N(T) = N(V)$ is finite dimensional. So the conclusion follows by [3, Proposition 3.14].

Proof of (3): As $VK = KV$, KV is self-adjoint. Hence $K^2 + 2Re(V^*K) = K^2 + 2VK$. Thus

$$
K^2 + 2VK = \begin{pmatrix} K_+^2 + 2K_+ & 0 & 0 \\ 0 & K_-^2 - 2K_- & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Since the $(1,1)$ entry of the above matrix is positive, to get the conclusion, it suffices to prove that the $(2,2)$ entry is positive. Clearly,

$K_-^2 - 2K_-$ is self-adjoint. Next, we show that $\sigma(K_-^2 - 2K_-)$ is positive. Let $\lambda \in \sigma(K_-)$. Then $\lambda \leq 0$ and $\lambda^2 - 2\lambda \in \sigma(K_-^2 - 2K_-)$. But $\lambda^2 - 2\lambda = \lambda(\lambda - 2) \geq 0$. Hence $K_-^2 - 2K_-$ is positive. □

Corollary 3.4. *Let $T = T^* \in \mathcal{AN}(H)$. Then $\sigma(T)$ is countable.*

Proof. Since $T = T_+ \oplus T_- \oplus T_0$ and all these operators T_+, T_- and T_0 are $\mathcal{AN}$ operators. We know that $\sigma(T_+), \sigma(T_0)$ are countable, as they are positive. Also, $-T_-$ is positive $\mathcal{AN}$-operator and hence $\sigma(T_-)$ is countable. Hence we can conclude that $\sigma(T) = \sigma(T_+) \cup \sigma(T_-) \cup \sigma(T_0)$ is countable. □

Next, we can get the structure of normal $\mathcal{AN}$-operators. Here we use a different approach to the one used in Theorem 3.3.

Proposition 3.5. *Let $T \in \mathcal{AN}(H)$ be normal with the polar decomposition $T = V|T|$. Then there exists a compact normal operator K, a finite rank normal operator $F \in \mathcal{B}(H)$ such that*

(1) *T has the representation:*

$$T = K - F + \alpha V \tag{3.1}$$

with $KF = 0$ and $F^*F \leq \alpha^2 I$

(2) *$K^*K + 2\alpha Re(V^*K) \geq 0$*

(3) *V, K, F commutes mutually*

(4) *if $\alpha > 0$, then $V \in \mathcal{AN}(H)$.*

Proof. Proof of (1): It is known that T is normal if and only if $V|T| = |T|V$. Since $|T| \in \mathcal{AN}(H)$, we have $|T| = K_1 - F_1 + \alpha I$, where $K_1 \in \mathcal{K}(H)$ is positive, $F_1 \in \mathcal{F}(H)$ is positive and $F_1 \leq \alpha I$.

First, we show that V is normal. We have $N(T^*) = N(T) = N(V)$. Hence

$$V^*V = P_{N(V)^\perp} = P_{N(T)^\perp} = P_{N(T^*)^\perp} = P_{\overline{R(T)}} = P_{R(V)} = VV^*.$$

So, $T = K - F + \alpha V$, where $K = VK_1$ and $F = VF_1$. Next, we show that K and F are normal. As T is normal, V commutes with $|T|$. Hence

$$V(K_1 - F_1) = (K_1 - F_1)V. \tag{3.2}$$

Since V commute with $K_1 - F_1$, it also commute with $(K_1 - F_1)^2$. But, $(K_1 - F_1)^2 = K_1^2 + F_1^2 = (K_1 + F_1)^2$. With this, we can conclude that $V(K_1 + F_1)^2 = (K_1 + F_1)^2 V$. Hence,

$$V(K_1 + F_1) = (K_1 + F_1)V. \tag{3.3}$$

Thus by Equations (3.2) and (3.3), we can conclude that $VK_1 = K_1V$ and $VF_1 = F_1V$. By the Fuglede's theorem we can conclude that

$V^*K_1 = K_1V^*$ and $V^*F_1 = F_1V^*$. Next,

$$K^*K = K_1V^*VK_1 = K_1VV^*K_1 = VK_1V^*K_1 = VK_1K_1V^* = KK^*.$$

With similar arguments we can show that F is normal.

Next, we show that $KF = 0$. Since V commute with K_1 and F_1, we have $KF = VK_1VF_1 = V^2K_1F_1 = 0$.

Finally, $F^*F = F_1V^*VF_1 \leq \|V\|^2F_1^2 \leq \alpha^2 I$.

Proof of (2): Using the relations $VK_1 = K_1V$ and $V^*K_1 = K_1V^*$, we get

$$\begin{aligned} K^*K + \alpha(V^*K + K^*V) &= K_1V^*VK_1 + \alpha(V^*VK_1 + K_1V^*V) \\ &= V^*V(K_1^2 + 2\alpha K_1) \\ &= P_{N(V)^\perp}(K_1^2 + 2\alpha K_1) \\ &= K_1^2 + 2\alpha K_1 \\ &\geq 0. \end{aligned}$$

In the fourth step of the above equations we have used the fact that $P_{N(V)^\perp}K_1 = P_{R(V)}K_1 = P_{R(|T|)}K_1 = K_1$.

Proof of (3): We have $VK = VVK_1 = VK_1V = KV$ and $VF = VVF_1 = VF_1V = FV$. Also, $KF = 0 = FK$.

Proof of (4): Note that by applying (2) of Proposition 2.7 to $|T|$, we can conclude that $N(|T|) = N(T) = N(V)$ is finite dimensional. Now the conclusion follows by [3, Proposition 3.14]. □

Corollary 3.6. *Let $T \in \mathcal{B}(H)$ be normal. Then $T \in \mathcal{AN}(H)$ if and only if $T^* \in \mathcal{AN}(H)$.*

Proof. We know that $T \in \mathcal{AN}(H)$ if and only if $T^*T \in \mathcal{AN}(H)$ by Corollary 2.10. Since $T^*T = TT^*$, by Corollary 2.10 again, it follows that $TT^* \in \mathcal{AN}(H)$ if and only if $T^* \in \mathcal{AN}(H)$. □

4. General case

In this section we prove the structure of absolutely norm attaining operators defined between two different Hilbert spaces.

Theorem 4.1. *Let $T \in \mathcal{AN}(H_1, H_2)$ with the polar decomposition $T = V|T|$. Then*

$$T = K - F + \alpha V,$$

where $K \in \mathcal{K}(H_1, H_2)$, $F \in \mathcal{F}(H_1, H_2)$ such that $K^*F = 0 = KF^*$ and $\alpha^2 I \geq F^*F$.

Proof. Since $|T| \in \mathcal{AN}(H_1)$ and positive, we have by Theorem 2.4, $|T| = K_1 - F_1 + \alpha I$, where the triple (K_1, F_1, α) satisfy conditions in

(2) of Theorem 2.4. Now, $T = K - F + \alpha V$, where $K = VK_1,\ F = VF_1$. Clearly,

$$\begin{aligned} K^*F = K_1V^*VF_1 = K_1P_{N(V)^\perp}F_1 &= K_1(I - P_{N(V)})F_1 \\ &= K_1F_1 - K_1P_{N(V)}F_1 \\ &= 0 \text{ (since } \ N(V) = N(|T|) \subseteq N(K_1)). \end{aligned}$$

Also, clearly, $KF^* = VK_1F_1V^* = 0$.

Finally, $F^*F = F_1V^*VF_1 \leq \|V^*V\|F_1^2 \leq F_1^2 \leq \alpha^2 I$. □

Proposition 4.2. *Let $T \in \mathcal{B}(H)$ and $U \in \mathcal{B}(H)$ be unitary such that $T^* = U^*TU$. Then $T \in \mathcal{AN}(H)$ if and only if $T^* \in \mathcal{AN}(H)$.*

Proof. This follows by [3, Theorem 3.5]. □

Next, we discuss a possible converse in the general case.

Theorem 4.3. *Let $K \in \mathcal{K}(H_1, H_2),\ F \in \mathcal{F}(H_1, H_2),\ \alpha \geq 0$ and $V \in \mathcal{B}(H_1, H_2)$ be a partial isometry. Further assume that*

(1) $V \in \mathcal{AN}(H_1, H_2)$

(2) $K^*K + \alpha(V^*K + K^*V) \geq 0$.

Then $T := K - F + \alpha V \in \mathcal{AN}(H_1, H_2)$.

Proof. If $\alpha = 0$, then $T \in \mathcal{K}(H_1, H_2)$. Hence $T \in \mathcal{AN}(H_1, H_2)$. Next assume that $\alpha > 0$. We prove this case by showing $T^*T \in \mathcal{AN}(H_1)$. By a simple calculation we can get $T^*T = \mathcal{K} - \mathcal{F} + \alpha^2 P_{N(V)^\perp}$, where,

$$\mathcal{K} = K^*K + \alpha(V^*K + K^*V), \quad \mathcal{F} = F^*F - F^*K - K^*F - \alpha(V^*F + F^*V).$$

Since $V \in \mathcal{AN}(H_1, H_2)$, either $N(V)$ or $N(V)^\perp$ is finite dimensional. If $N(V)^\perp$ is finite dimensional, then $T^*T \in \mathcal{K}(H_1)$. Hence $T \in \mathcal{K}(H_1, H_2)$.

If $N(V)$ is finite dimensional, then $T^*T = \mathcal{K} - (\mathcal{F} - \alpha^2 P_{N(V)}) + \alpha^2 I$. Note that the operator $\mathcal{F} - \alpha^2 P_{N(V)}$ is a finite rank self-adjoint operator. Hence $T^*T \in \mathcal{AN}(H_1)$ by Theorem 2.1. Now the conclusion follows by Corollary 2.10. □

Corollary 4.4. *Suppose that $K \in \mathcal{K}(K),\ F \in \mathcal{F}(H)$ are normal and $V \in \mathcal{B}(H)$ is a normal partial isometry such that V, F, K commute mutually. Let $\alpha \geq 0$. Then*

(1) *$T := K - F + \alpha V$ is normal and*

(2) *if $K^*K + 2\alpha V^*K \geq 0$ and $V \in \mathcal{AN}(H)$, then $T \in \mathcal{AN}(H)$.*

Proof. To prove (1) we observe that if A and B are commuting normal operators, then $A + B$ is normal (see [13, Page 342, Exercise 12] for details). By this observation it follows that T is normal.

To prove (2), since $VK = KV$, by Fuglede's theorem [13, Page 315], $V^*K = KV^*$. With this observation and Theorem 4.3, the conclusion follows. □

Corollary 4.5. *Suppose that $K \in \mathcal{K}(H)$, $F \in \mathcal{F}(H)$ are self-adjoint and $V \in \mathcal{B}(H)$ is a self-adjoint, partial isometry such that*

(a) $V \in \mathcal{AN}(H)$
(b) $K^2 + 2\alpha(VK) \geq 0$.

Then $T := K - F + \alpha V$ is self-adjoint and $\mathcal{AN}$-operator.

Proof. The proof directly follows by Theorem 4.3. □

Definition 4.6. [4, page 349] Let $T \in \mathcal{B}(H_1, H_2)$. Then T is called left *semi-Fredholm* if there exists a $B \in \mathcal{B}(H_2, H_1)$ and $K \in \mathcal{K}(H_1)$ such that $BT = K + I$ and *right semi-Fredholm* if there exists a $A \in \mathcal{B}(H_2, H_1)$ and $K' \in \mathcal{K}(H_2)$ such that $TA = K' + I$.

If T is both left semi-Fredholm and right semi-Fredholm, then T is called Fredholm.

Remark 4.7. Note that T is left semi-Fredholm if and only if T^* is right semi-Fredholm (see [4, section 2, page 349] for details).

Corollary 4.8. *Let $T \in \mathcal{AN}(H_1, H_2)$ but not compact. Then T is left-semi-Fredholm.*

Proof. Let $T = V|T|$ be the polar decomposition of T. Then $|T| = V^*T$. As, $|T| \in \mathcal{AN}(H_1)$, by Theorem 2.4, there exists a triple (K, F, α) satisfying conditions in Theorem 2.4, such that $V^*T = K - F + \alpha I$. Let $K^{'} = K - F$. Then $V^*T = K^{'} + \alpha I$. By Definition 4.6, it follows that T is left-semi-Fredholm. □

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