Nullspace Property for Optimality of Minimum Frame Angle Under Invertible Linear Operators

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Abstract—**Frames with a large minimum angle between any two distinct frame vectors are desirable in many present day applications. For a unit norm frame, the absolute value of the cosine of the minimum frame angle is also known as coherence. Two frames are equivalent if one can be obtained from the other via left action of an invertible linear operator. Frame angles can change under the action of a linear operator. Most of the existing works solve different optimization problems to find an optimal linear operator that maximizes the minimal frame angle (in other words, minimizes the coherence). In the present work, nevertheless, we consider the question: Is it always possible to find an equivalent frame with smaller coherence for a given frame?. In this paper, we derive properties of the initial unit norm frame that can ensure an equivalent frame with strictly larger minimal frame angle compared to the initial one. It turns out that the nullspace property of a certain matrix obtained from the initial frame can guarantee such an equivalent frame. We also present the numerical results that support our theoretical claims.**

Index Terms—**Minimum frame angle, coherence, preconditioning, compressed sensing, semidefinite programming.**

I. INTRODUCTION

FRAME theory has applications in fields such as signal processing, sparse representation theory and operator theory. A finite frame can be represented as a matrix of full row rank. The coherence of a frame is defined as the largest absolute normalized inner product between two distinct frame vectors. A finite frame with small coherence is said to be an incoherent frame. For a fixed number of elements, frames with the smallest coherence are called Grassmannian frames [12]. Grassmanian frames attaining the Welch bound are known as equiangular tight frames (ETFs) [7]. Incoherent frames play a significant role due to their ability in providing sparse representations.

The field of sparse representation theory, popularly known as compressed sensing (CS) [5], [8], recovers a sparse signal from a few of its linear measurements. Performance of several sparse recovery algorithms such as basis pursuit (BP) and orthogonal matching pursuit (OMP) depends on the coherence of the underlying frame. Frames that satisfy the restricted isometry property (RIP) [3], [10] are known to allow for exact recovery of sparse signals from a few of their linear measurements. The coherence of a frame can assert RIP up to certain order [6].

Two linear systems of equations provided by two frames are equivalent if the underlying frames are equivalent. However, the sparse recovery properties like RIP and coherence of equivalent frames can be significantly different [11], [15]. Consequently, the performances of sparse recovery algorithms can be different. Several methods exist in the literature [1], [4], [9], [5], [16] for finding an equivalent frame with optimal coherence.

The results reported, for example, in [1], [4], [9] proposed ways of finding the incoherent frames and their relevance to applications. It was shown in [16] that the incoherence obtained via preconditioning results in significant improvement in tomographic image reconstruction.

Although these methods work well in practice, they do not possess theoretical guarantees for reduction in coherence. In this work, however, we consider the question: "For a given frame, when is it possible to find an equivalent frame with smaller coherence?." In particular, in contrast to existing results, we show and demonstrate numerically that it is not always possible to obtain a strictly incoherent and equivalent frame. In light of the present contribution, given a frame, one can verify if there is any scope for obtaining its coherence improvement as low coherence has a bearing on the quality of the underlying signal to be reconstructed and/or number of measurements to be used in such applications as tomographic reconstruction [16].

The main objective of the present work is to derive sufficient conditions on a frame that can ensure an equivalent frame possessing smaller coherence. We show that the existence of such an equivalent frame can be ascertained by checking for in-feasibility of a linear system of equations. The null space property of a certain matrix obtained from the initial frame ensures the existence of an equivalent frame with a strictly smaller coherence. The main contributions of the paper are summarized below:

- We derive the sufficient conditions that can guarantee existence of an equivalent frame having smaller coherence compared to the initial frame.
- We present numerical results that validate our theoretical analysis.

II. BASICS OF FRAME THEORY

A. Frame Theory

A family of vectors $\{\phi_i\}_{i=1}^M$ in $\mathbb{R}^N$ is called a frame [2] for $\mathbb{R}^N$, if there exist constants $0 < A \leq B < \infty$ such that

$$A \|z\|^2 \leq \sum_{i=1}^M |\langle z, \phi_i \rangle|^2 \leq B \|z\|^2, \forall z \in \mathbb{R}^N,$$

where A and B are called the lower and upper frame bounds respectively. If $A = B$, then $\{\phi_i\}_{i=1}^M$ is an $A-$tight frame. If there exists a constant d such that $|\langle \phi_i, \phi_j \rangle| = d$, for $1 \leq i < j \leq M$, then $\{\phi_i\}_{i=1}^M$ is an equiangular frame. If there exits a constant c such that $\|\phi_i\|_2 = c$ for all $i = 1, 2, \ldots, M$, then $\{\phi_i\}_{i=1}^M$ is an equal norm frame. If $c = 1$, then it is called a unit norm frame. If a frame is both unit norm and tight, it is called a unit norm tight frame (UNTF). If a frame is both UNTF and equiangular, it is called an equiangular tight frame (ETF). The coherence of a frame Φ is given by

$$\mu(\Phi) = \max_{1 \leq i,j \leq M, \, i \neq j} \frac{|\phi_i^T \phi_j|}{\|\phi_i\|_2 \|\phi_j\|_2}.$$

Coherence based techniques are used in establishing the guaranteed recovery of sparse signals via Orthogonal Matching Pursuit (OMP) or Basis Pursuit (BP), as summarized by the following result [13].

Theorem II.1: An arbitrary $k-$sparse signal x can be uniquely recovered using OMP and BP, provided

$$k < \frac{1}{2}\left(1 + \frac{1}{\mu(\Phi)}\right). \tag{1}$$

If G is a nonsingular matrix, then the system $Gy = G\Phi x$ is equivalent to $y = \Phi x$. The bound in (1) then suggests that if $\mu(G\Phi) < \mu(\Phi)$ both BP and OMP have better performance guarantees when applied on the equivalent system $Gy = G\Phi x$.

III. MAIN RESULTS

In this section, we present the properties of an initial frame that can ensure strict fall in coherence under the left multiplication of an invertible linear operator. Our main results concerning the sufficient conditions on the initial frames can be given by the following theorem.

Theorem III.1: For a given unit norm frame $\Phi_{N \times M}$ for $\mathbb{R}^N$ with coherence $\mu(\Phi)$, let ϕ_i denote the ith column of Φ and suppose

$$D_{I_N}^+ := \{(i,j) : \phi_i^T \phi_j = \mu(\Phi)\}$$

and

$$D_{I_N}^- := \{(i,j) : \phi_i^T \phi_j = -\mu(\Phi)\}.$$

Consider the matrix $\Psi_{N^2 \times (M + |D_{I_N}^+| + |D_{I_N}^-|)} =$

$$\left[\left(\text{vec}(\phi_i \phi_i^T)\right)_{i=1}^M \left(\text{vec}(\phi_{ij}')\right)_{(i,j) \in D_{I_N}^+} \right.$$
$$\left. - \left(\text{vec}(\phi_{ij}')\right)_{(i,j) \in D_{I_N}^-} \right],$$

where $\phi_{ij}' := \frac{\phi_i \phi_j^T + \phi_j \phi_i^T}{2}$, and the 'vec' operation on a matrix of size $N \times M$ produces a vector of length NM by stacking the

columns one below the other vertically. If there does not exist a vector $r \in \mathbb{R}^{M + |D_{I_N}^+| + |D_{I_N}^-|}$ in the nullspace of Ψ satisfying

$$\sum_{k=1}^M r_k = -\mu_\Phi$$

and

$$\sum_{k=M+1}^{M + |D_{I_N}^+| + |D_{I_N}^-|} r_k = 1,$$

then there exits an invertible operator G such that $\mu(G\Phi) < \mu(\Phi)$.

Proof: Let S be the set of invertible operators G such that $G\Phi$ is a unit norm frame for $\mathbb{R}^N$, that is,

$$S = \{G \in \mathbb{R}^{N \times N} : |G| \neq 0, \|G\Phi_i\|_2 = 1, \forall i = 1, 2, \ldots, M\},$$

where $|G|$ denotes the determinant of G. It can be noted that $S \neq \emptyset$ as $I_{N \times N} \in S$, where $I_{N \times N}$ is the identity matrix. Therefore, in order to show that there exists an invertible operator $G \in S$ such that $\mu_{G\Phi} < \mu_\Phi$, it is enough to show that $I_{N \times N}$ is not an optimal solution to the following optimization problem:

$$C_0 : \underset{G \in S}{\arg \min} \quad \max_{i \neq j} |\langle G\phi_i, G\phi_j \rangle|.$$

An equivalent formulation of C_0 is

$$\underset{X}{\arg \min} \quad \max_{i \neq j} |\phi_i^T X \phi_j|$$

$$\text{subject to} \quad \phi_i^T X \phi_i = 1, \quad \forall i = 1, \ldots, M.$$
$$X \succ 0,$$

where $X = G^T G$ and $X \succ 0$ denotes that X is positive definite. The advantage of the equivalent formulation of C_0 is that the constraints are linear in X and the objective function is convex in X. Since the constraint set

$$S_0 = \{X \succ 0 : \phi_i^T X \phi_i = 1, \quad \forall i = 1, \ldots, M\}$$

is convex but not closed, we consider

$$S_1 = \{X \succeq 0 : \phi_i^T X \phi_i = 1, \quad \forall i = 1, \ldots, M\},$$

where $X \succeq 0$ implies that X is positive semi-definite with the corresponding convex optimization problem,

$$C_0' : \underset{X}{\arg \min} \quad \max_{i \neq j} |\phi_i^T X \phi_j|$$

$$\text{subject to} \quad X \in S_1.$$

Adding slack and surplus variables $p_{ij} \geq 0$ and $q_{ij} \geq 0$ respectively, one may obtain an equivalent formulation of C_0':

$$\underset{X, q, p_{ij}, q_{ij}}{\max} \quad (-q)$$

$$\text{subject to} \quad \phi_i^T X \phi_i = 1, \forall i = 1, \ldots, M;$$

$$\phi_i^T X \phi_j + p_{ij} - q = 0, \forall 1 \leq i < j \leq M;$$

$$C_1 : \qquad - \phi_i^T X \phi_j + q_{ij} - q = 0, \forall 1 \leq i < j \leq M;$$

$$X \succeq 0$$

$$q \geq 0$$

$$p_{ij} \geq 0, \; \forall 1 \leq i < j \leq M;$$

$$q_{ij} \geq 0, \; \forall 1 \leq i < j \leq M.$$

Using $M' = \frac{M(M-1)}{2}$, let $\mathbf{0}$ denote the zero matrix of size $N \times N$, $\mathbf{0}'$ a square zero matrix of size $M' \times M'$, P a diagonal matrix of size $M' \times M'$ consisting of p_{ij} as diagonal elements, Q a diagonal matrix of size $M' \times M'$ containing q_{ij} as diagonal elements. Finally, let $\mathbf{1}_{ij}$ be the diagonal matrix of size $M' \times M'$ whose diagonal entries are indexed by arranging the tuples (i,j) in lexicographic order for $1 \le i < j \le M$ so that it contains 1 at the $(i,j)-$th diagonal element and zero elsewhere. For simplicity in notation, we consider $\phi'_{ij} = \frac{\phi_i \phi_j^T + \phi_j \phi_i^T}{2}$ and define the following block matrices:

$$F_0 = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{0}' & 0 & 0 \\ 0 & 0 & \mathbf{0}' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; F_{ii} = \begin{pmatrix} \phi_i \phi_i^T & 0 & 0 & 0 \\ 0 & \mathbf{0}' & 0 & 0 \\ 0 & 0 & \mathbf{0}' & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F_{ij} = \begin{pmatrix} \phi'_{ij} & 0 & 0 & 0 \\ 0 & \mathbf{1}_{ij} & 0 & 0 \\ 0 & 0 & \mathbf{0}' & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; Y = \begin{pmatrix} X & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

$$F_{ji} = \begin{pmatrix} -\phi'_{ij} & 0 & 0 & 0 \\ 0 & \mathbf{0}' & 0 & 0 \\ 0 & 0 & \mathbf{1}_{ij} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is easy to check that, for $1 \le i < j \le M$, F_{ii}, F_{ij}, F_{ji} and F_0 are symmetric. Using these matrices, we reformulate C_1 in a standard Semi-definite Programming (SDP) [14] form as

$$SDP_{\mathbf{C}_1}: \quad \begin{aligned} \max_Y \quad & -Tr(F_0 Y) \\ \text{subject to} \quad & Tr(F_{ii} Y) = 1, \quad \forall \; i = 1, \dots, M; \\ & Tr(F_{ij} Y) = 0, \quad \forall \; 1 \le i < j \le M; \\ & Tr(F_{ji} Y) = 0, \quad \forall \; 1 \le i < j \le M, \\ & Y \succeq 0, \end{aligned}$$

where $Tr(A)$ represents trace of the matrix A and $Y \succeq 0$ implies that Y is positive semi-definite, that is, $\zeta^T Y \zeta \ge 0$ for all $\zeta \in \mathbb{R}^{M^2-M+N+1}$. The dual of $SDP_{\mathbf{C}_1}$ is given by

$$SDPD_{\mathbf{C}_1}: \quad \begin{aligned} \min_{z=\{z_{ij}\}_{i,j=1}^M} \quad & c^T z \\ \text{subject to} \quad & F_0 + \sum_{i=1}^M z_{ii} F_{ii} + \sum_{i,j=1, i<j}^M z_{ij} F_{ij} + \\ & \sum_{i,j=1, i<j}^M z_{ji} F_{ji} \succeq \mathbf{0}'', \end{aligned}$$

where $\mathbf{0}''$ is a square zero matrix of size $M'' = M^2 - M + N + 1$ and $c = \{c_i\}_{i=1}^{M^2} \in \mathbb{R}^{M^2}$, where c_i is 1 for $i = 1, 2, \dots, M$ and 0 for $i = M+1, \dots, M^2$.

It is easy to check that, if X is the identity matrix, $p_{ij} = 1 - \phi_i^T \phi_j$, $q_{ij} = 1 + \phi_i^T \phi_j$ and $q = 1$, then Y becomes a strict feasible solution of $SDP_{\mathbf{C}_1}$. Similarly, one can verify that with $z_{ii} = 1$ and $z_{ij} = z_{ji} = \frac{1}{M^2}$, Z becomes a strict feasible solution of $SDPD_{\mathbf{C}_1}$. Since both primal and dual have strict feasible

solutions, by strong duality (Theorem 3.1 in [14]), optimal values of primal and dual optimization problems coincide with each other. Consequently, the duality gap is zero for any optimal pair (Y^*, Z^*), where Y^* is an optimal solution for $SDP_{\mathbf{C}_1}$ and Z^* is an optimal solution for $SDPD_{\mathbf{C}_1}$. The duality gap being zero implies that $0 = C_{M''}^T Z^* + Tr(F_0 Y) = \sum_{i=1}^M z_{ii}^* + q^*$ which implies further that $q^* = -\sum_{i=1}^M z_{ii}^*$.

The standard optimality condition (Equation (33) in [14]) concerning the primal and dual solutions can be written as

$$Tr(F_{ii} Y^*) = 1, \quad \forall \; i = 1, \dots, M;$$
$$Tr(F_{ij} Y^*) = 0, \quad \forall \; 1 \le i < j \le M;$$
$$Tr(F_{ji} Y^*) = 0, \quad \forall \; 1 \le i < j \le M,$$
$$Y \succeq 0.$$

and

$$Y^* \left(F_0 + \sum_{i=1}^M z_{ii}^* F_{ii} + \sum_{i,j=1, i<j}^M z_{ij}^* F_{ij} + \sum_{i,j=1, i<j}^M z_{ji}^* F_{ji} \right)$$
$$= \mathbf{0}'',$$
$$(c^*)^T z^* = q^*.$$

The above condition results in the following equations:

1) $Tr\left(X^*(\phi_i \phi_i^T) \right) = 1, \quad \forall \; i = 1, \dots, M$
2) $Tr(X^* \phi'_{ij}) + p_{ij}^* - q^* = 0, \quad \forall \; 1 \le i < j \le M$
3) $Tr(-X^* \phi'_{ij}) + q_{ij}^* - q^* = 0, \quad \forall \; 1 \le i < j \le M$
4) $X^* \left(\sum_{i=1}^M z_{ii}^* \phi_i \phi_i^T + \sum_{i,j=1, i<j}^M (z_{ij}^* - z_{ji}^*) \phi'_{ij} \right) = 0$
5) $z_{ij}^* p_{ij}^* = 0$
6) $z_{ji}^* q_{ij}^* = 0$
7) $q^* \left(1 - \sum_{i,j=1, i<j}^M (z_{ij}^* + z_{ji}^*) \right) = 0$
8) $-\sum_{i=1}^M z_{ii}^* = q^*$.

If X^* is assumed to be positive definite, then the fourth equality above reduces to

$$\sum_{i=1}^M z_{ii}^* \phi_i \phi_i^T + \sum_{i,j=1, i<j}^M (z_{ij}^* - z_{ji}^*) \phi'_{ij} = 0. \tag{2}$$

For a positive semi-definite matrix $X = G^T G$, let $D^+(X, q)$ and $D^-(X, q)$ be the sets of tuples of indices for which the corresponding entry of $\Phi^T X \Phi$ (equivalently the inner-product between two corresponding columns of $G\Phi$) is equal to q and $-q$ respectively, that is,

$$D^+(X, q) = \{(i,j) : \phi_i^T X \phi_j = q\}$$

and

$$D^-(X, q) = \{(i,j) : \phi_i^T X \phi_j = -q\}.$$

It is clear that

$$p_{ij}^* = 0 \quad \text{if and only if} \quad (i,j) \in D^+(X^*, q^*)$$

and

$$q_{ij}^* = 0, \quad \text{if and only if} \quad (i,j) \in D^-(X^*, q^*).$$

From the definition of $D^+(X^*, q^*)$ and $D^-(X^*, q^*)$, it follows that

$$z_{ij}^* = 0, \quad \text{if} \quad (i,j) \notin D^+(X^*, q^*) \tag{3}$$

and

$$z_{ji}^* = 0, \quad \text{if} \quad (i,j) \notin D^-(X^*, q^*). \tag{4}$$

Since $q^* > 0$,

$$
\begin{aligned}
1 &= \sum_{i,j=1, i<j}^M (z_{ij}^* + z_{ji}^*) \\
&= \sum_{(i,j) \in D^+(X^*, q^*)} z_{ij}^* + \sum_{(i,j) \in D^-(X^*, q^*)} z_{ji}^*.
\end{aligned} \tag{5}
$$

Therefore, a positive definite matrix X^* is optimal if and only if (2), (3), (4), (5) and $-\sum_{i=1}^M z_{ii}^* = q^*$ are satisfied. In other words, if there exists z^* satisfying the above equations then X^* is optimal. Hence, if $X^* = I_{N \times N}$ is not an optimal solution there do not exist scalars $\{\tilde{z}_{ij}\}_{i,j=1}^M$ satisfying the following conditions

1) $\tilde{z}_{ij} = 0$ for $(i,j) \notin D^+(\mathbf{I}_N, \mu_\Phi)$,
2) $\tilde{z}_{ji} = 0$ for $(i,j) \notin D^-(\mathbf{I}_N, \mu_\Phi)$,
3) $\sum_{i,j=1, i<j}^M (\tilde{z}_{ij} + \tilde{z}_{ji}) = 1$,
4) $\sum_{i=1}^M \tilde{z}_{ii} \phi_i \phi_i^T + \sum_{i,j=1, i<j}^M (\tilde{z}_{ij} - \tilde{z}_{ji}) \phi_{ij}' = 0$,
5) $-\sum_{i=1}^M \tilde{z}_{ii} = \mu(\Phi)$.

The above conditions can be written as a linear system of equations $\tilde{\Psi}\tilde{z} = y$, where $\tilde{\Psi}_{(N^2+2) \times (M+|D_{I_N}^+|+|D_{I_N}^-|)} =$

$$
\begin{bmatrix}
\left(vec(\phi_i \phi_i^T)\right)_{i=1}^M & \left(vec(\phi_{ij}')\right)_{(i,j) \in D_{I_N}^+} & -\left(vec(\phi_{ij}')\right)_{(i,j) \in D_{I_N}^-} \\
0_{1 \times M} & 1_{1 \times |D_{I_N}^+|} & 1_{1 \times |D_{I_N}^-|} \\
1_{1 \times M} & 0_{1 \times |D_{I_N}^+|} & 0_{1 \times |D_{I_N}^-|}
\end{bmatrix},
$$

$\tilde{z}_{(M+|D_{I_N}^+|+|D_{I_N}^-|) \times 1} =$

$$\left[(\tilde{z}_{ii})_{i=1}^M \quad (\tilde{z}_{ij})_{(i,j) \in D_{I_N}^+} \quad (\tilde{z}_{ji})_{(i,j) \in D_{I_N}^-}\right]^T,$$

and $y_{(N^2+2) \times 1} = [0_{N^2 \times 1} \quad 1_{1 \times 1} \quad -\mu(\Phi)]^T$.

Therefore, the infeasiblity of the linear system $\tilde{\Psi}\tilde{z} = y$ guarantees the existence of an operator $\hat{G}$ such that $\hat{G}\Phi$ is a unit norm frame and $\mu(\hat{G}\Phi) < \mu(\Phi)$. Despite this, there is no guarantee that $\hat{G}$ is positive definite. Nevertheless, there exist nonsingular matrices G_n so that $G_n \to \hat{G}$ in the Frobenious norm. Since coherence $\mu(G\Phi)$ is a continuous function of G in the Frobenious norm, we have $\mu(G_n\Phi) \to \mu(\hat{G}\Phi)$. This ensures that there exists a non-singular matrix G for which $\mu(G\Phi) < \mu(\Phi)$. ∎

The following corollary follows from the main theorem III.1.

Corollary III.2: If Ψ has a trivial nullspace, then there exits an invertible operator $\hat{G}$ such that $\mu(\hat{G}\Phi) < \mu(\Phi)$.

Proof: The proof follows from the fact that if Ψ has a trivial nullspace then the two equations given in Theorem III.1 can not be satisfied. ∎

So far, we discussed our theoretical findings. In the next section, we present numerical results in support of our analytical results.

IV. NUMERICAL OBSERVATIONS

In this section, we present the effect of coherence on random Gaussian matrices by invertable linear operators G, obtained by solving the C_0' problem (See Section III). To begin with, we

TABLE I
SOLVING C_0' FOR GAUSSIAN RANDOM MATRICES WITH ROW SIZE 10 AND COLUMN SIZE INCREMENTED BY 10 STARTING WITH 20

$N \times M$	$\mu(\Phi)$	$M + |D_{I_N}^+| + |D_{I_N}^-|$	rank (Ψ)	$\mu(G\Phi)$
10×20	0.7225	22	22	0.4968
10×30	0.7468	32	32	0.6234
10×40	0.8738	42	42	0.7541
10×50	0.8509	52	52	0.8261
10×60	0.8765	62	56	0.8765
10×70	0.8626	72	56	0.8626

TABLE II
APPLYING OPTIMIZATION METHOD IN [9] ON GAUSSIAN RANDOM MATRICES WITH ROW SIZE 300 AND DIFFERENT COLUMN SIZES STARTING WITH 610

$N \times M$	$\mu(\Phi)$	$M + |D_{I_N}^+| + |D_{I_N}^-|$	rank(Ψ)	$\mu(G\Phi)$
300×610	0.2562	612	612	0.1985
300×810	0.2742	812	812	0.2219
300×1010	0.2898	1012	1012	0.2368
300×1210	0.2778	1212	1212	0.2648
300×1410	0.2759	1412	1412	0.2536
300×1510	0.2731	1512	1512	0.2649

considered random Gaussian matrices $\Phi \in \mathbb{R}^{N \times M}$ for different row sizes N along with varying value for the column size M.

As examples, we fixed row size as 10, while varying the column size, and generated the Table I. From Table I, for $M \leq 50$, it may be noted that the rank of the corresponding matrix Ψ (as defined in Theorem III.1) and $M + |D_{I_N}^+| + |D_{I_N}^-|$, the column size of Ψ, are the same. As a result, Ψ has trivial nullspace. Consequently, from Corollary III.2, one can expect a strict fall in coherence. In Table I, we observe the similar behaviour on the coherence as predicted by Corollary III.2, that is, for $M \leq 50$, $\mu(G\Phi)$ is strictly smaller than $\mu(\Phi)$. For $M > 50$, $M + |D_{I_N}^+| + |D_{I_N}^-|$ becomes strictly greater than rank of Ψ and we observe from Table I that the coherence remains unchanged.

For a given frame Φ, the authors in [9], proposed a method to find an invertible operator G for which the associated gram matrix of the transformed frame becomes close to the identity matrix. In other words, the transformed frame becomes close to a unit norm frame and has small coherence. For the frames obtained via solving optimization problem described in [9], we observe similar behaviour on the coherence provided in Table II as predicted by Corollary III.2. Therefore, we can justify the fall in coherence as described in [9] via proposed null space property.

V. CONCLUDING REMARKS

In the present work, we derived properties of an initial frame that ensure strict fall in coherence via left multiplication by an invertible linear operator. It turns out that the infeasibilty of a linear system of equations obtained from the initial frame results in an equivalent frame with a larger minimum frame angle. In particular, if a certain matrix obtained from initial frame possesses trivial nullspace, then there exists an equivalent frame with strictly smaller coherence. The numerical results also support our theoretical analysis.

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