NOTES ON ATKIN-LEHNER THEORY FOR DRINFELD MODULAR FORMS

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ABSTRACT. In this article, we settle a part of the Conjecture by Bandini and Valentino ([BV19a]) for $S_{k,l}(\Gamma_0(T))$ when dim $S_{k,l}(\mathrm{GL}_2(A)) \leq 2$. Then, we frame this conjecture for prime, higher levels, and provide some evidence in favor of it. For any square-free level $\mathfrak{n}$, we define oldforms $S_{k,l}^{\mathrm{old}}(\Gamma_0(\mathfrak{n}))$, newforms $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$, and investigate their properties. These properties depend on the commutativity of the (partial) Atkin-Lehner operators with the $U_{\mathfrak{p}}$-operators. Finally, we show that the set of all $U_{\mathfrak{p}}$-operators are simultaneously diagonalizable on $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$.

1. INTRODUCTION

The theory of oldforms and newforms is a well-understood area in the theory of classical modular forms. Certain properties of modular forms heavily depend on whether they belong to oldforms or newforms. For example, the space of newforms has a basis consisting of normalized eigenforms for all the Hecke operators. In fact, the Fourier coefficients of these normalized eigenforms generate a number field. To the best of author's knowledge, the analogues theory of oldforms and newforms is not much known for Drinfeld modular forms.

In this article, we propose a definition of oldforms, newforms for Drinfeld modular forms of square-free level. We justify these definitions by showing that these spaces are invariant under the action of the Hecke operators. The proof requires the commutativity of the (partial) Atkin-Lehner operators with the $U_{\mathfrak{p}}$-operators and certain properties of the space of $\mathfrak{p}$-oldforms and $\mathfrak{p}$-newforms.

In a series of papers (cf. [BV19], [BV19a], [BV20], [Val22]), Bandini and Valentino have defined $\mathfrak{p}$-oldforms, $\mathfrak{p}$-newforms and studied some of their properties. In [BV19], the authors defined T-oldforms $S_{k,l}^{T-\mathrm{old}}(\Gamma_0(T))$, T-newforms $S_{k,l}^{T-\mathrm{new}}(\Gamma_0(T))$ for $\mathfrak{p} = (T)$. In [BV19a], a sequel to [BV19], they have made the following conjecture:

Conjecture 1.1. ([BV19a, Conjecture 1.1] for $\Gamma_0(T)$)
(i) $\ker(T_T) = 0$ where T_T is acting on $S_{k,l}(\mathrm{GL}_2(A))$,
(ii) $S_{k,l}(\Gamma_0(T)) = S_{k,l}^{T-\mathrm{old}}(\Gamma_0(T)) \oplus S_{k,l}^{T-\mathrm{new}}(\Gamma_0(T))$,
(iii) U_T is diagonalizable on $S_{k,l}(\Gamma_0(T))$.

In [BV19a], [BV22], the authors proved Conjecture 1.1 in some special cases, using harmonic cocycles, the trace maps Tr and Tr′, and the linear algebra interpretation of the Hecke operators $T_{\mathfrak{p}}$ and $U_{\mathfrak{p}}$-operators. In this article, by studying the action of the T_T-operators on the Fourier coefficients of Drinfeld modular forms, we prove:

Theorem 1.2 (Theorem 4.6, Theorem 4.7)**.** *If* $\dim S_{k,l}(\mathrm{GL}_2(A)) \leq 1$*, then Conjecture 1.1 is true for* $S_{k,l}(\Gamma_0(T))$*. Furthermore, if* $\dim S_{k,l}(\mathrm{GL}_2(A)) = 2$*, then* $S_{k,l}(\Gamma_0(T)) = S_{k,l}^{T-\mathrm{old}}(\Gamma_0(T)) \oplus S_{k,l}^{T-\mathrm{new}}(\Gamma_0(T))$ *holds.*

Our methods in the proof of Theorem 4.6 and Theorem 4.7 are completely different from that of [BV19a], [BV22]. Our methods are based on the analysis of the Fourier coefficients of the image of an element via the Hecke operator T_T. We are very optimistic that our methods are suitable for generalizations, i.e., when $\dim S_{k,l}(\mathrm{GL}_2(A)) \geq 3$.

In a continuation work ([BV20]), for any prime ideal $\mathfrak{p}$, the authors extended the definition of $\mathfrak{p}$-oldforms, $\mathfrak{p}$-newforms to level $\mathfrak{p}$, level $\mathfrak{pm}$ with $\mathfrak{p} \nmid \mathfrak{m}$. So, it is quite natural to understand Conjecture 1.1 for level $\mathfrak{p}$, level $\mathfrak{pm}$ with $\mathfrak{p} \nmid \mathfrak{m}$. In this article, we frame it as a question (cf. Question 4.3) and provide some evidences in favor of it.

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First, we generalize the results of [BV19a] for $\mathfrak{p} = (T)$ to any arbitrary prime ideal $\mathfrak{p}$ (cf. Proposition 4.10). This implies that Question 4.3 has an affirmative answer in these cases. Then, we exhibit some cases where Question 4.3 for the level $\mathfrak{p}\mathfrak{m}$ is true (cf. Proposition 4.11). Here, we bring a word of caution. If $\mathfrak{m} \neq A$, we show that the direct sum decomposition in Question 4.3(2) may fail when $l = 1$ (cf. Proposition 4.13, Remark 4.14). More precisely, we exhibit non-zero Drinfeld cusp forms which are both $\mathfrak{p}$-oldforms and $\mathfrak{p}$-newforms. We believe that this is the only case where it may fail, and in fact, it may serve as a guiding example in future works.

In the final section, we propose a definition of oldforms $S_{k,l}^{\mathrm{old}}(\Gamma_0(\mathfrak{n}))$ and newforms $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ for Drinfeld modular forms of square-free level $\mathfrak{n}$. In fact, we justify our definition by showing that these spaces are invariant under the action of the Hecke operators (cf. Theorem 5.5). This requires the commutativity of the (partial) Atkin-Lehner operators with the $T_\mathfrak{p}$ and $U_\mathfrak{p}$-operators. For the $T_\mathfrak{p}$-operators, this is exactly Theorem 1.1 of [Val22]. Then, we prove an analogous result for the $U_\mathfrak{p}$-operators (cf. Theorem 5.4). Finally, we prove that the $U_\mathfrak{p}$-operators are simultaneously diagonalizable on $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ (cf. Corollary 5.6).

Notations: Throughout the article, we use the following notations:

- Let p be an odd prime number and $q = p^r$ for some $r \in \mathbb{N}$.
- Let $k \in \mathbb{N}$ and $l \in \mathbb{Z}/(q-1)\mathbb{Z}$ such that $k \equiv 2l \pmod{q-1}$. Let $0 \leq l \leq q-2$ be a lift of $l \in \mathbb{Z}/(q-1)\mathbb{Z}$. By abuse of notation, we write l for the integer as well as its class.

Let $\mathbb{F}_q$ denote the finite field of order q. Set $A := \mathbb{F}_q[T]$, $K := \mathbb{F}_q(T)$. Let $K_\infty = \mathbb{F}_q((\frac{1}{T}))$ be the completion of K with respect to the infinite place ∞ (corresponding to $\frac{1}{T}$-adic valuation), and denote by $C := \widehat{\overline{K_\infty}}$, the completion of $\overline{K_\infty}$. Let $\mathfrak{p} = (P)$ denote a prime ideal of A with a monic irreducible polynomial P.

An overview of the article. The article is organized as follows. In §2, we recall some basic theory of Drinfeld modular forms. In §3, we introduce certain important operators and study the inter-relations between them. In §4 we prove Theorem 1.2 and study the validity of Question 4.3 for prime, higher levels. In §5, we define oldforms, newforms and show that they are invariant under the action of the Hecke operators.

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2. Basic theory of Drinfeld modular forms

In this section, we recall some basic theory of Drinfeld modular forms (cf. [Gos80], [Gos80a], [Gek88], [GR96] for more details).

Let $L = \tilde{\pi}A \subseteq C$ be the A-lattice of rank 1 corresponding to the rank 1 Drinfeld module (which is also called Carlitz module) $\rho_T = TX + X^q$, where $\tilde{\pi} \in K_\infty(\sqrt[q-1]{-T})$ is defined up to a $(q-1)$-th root of unity. The Drinfeld upper half-plane $\Omega = C - K_\infty$, which is analogue to the complex upper half-plane, has a rigid analytic structure. The group $\mathrm{GL}_2(K_\infty)$ acts on Ω via fractional linear transformations.

Definition 2.1. Let $k \in \mathbb{N}$, $l \in \mathbb{Z}/(q-1)\mathbb{Z}$ and $f : \Omega \to C$ be a rigid holomorphic function on Ω. For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\infty)$, we define the slash operator $|_{k,l}\gamma$ on f by

$$f|_{k,l}\gamma := (\det\gamma)^l (cz+d)^{-k} f(\gamma z). \tag{2.1}$$

For an ideal $\mathfrak{n} \subseteq A$, let $\Gamma_0(\mathfrak{n})$ denote the congruence subgroup $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A) : c \in \mathfrak{n}\}$. Now, we define a Drinfeld modular form of weight k, type l for $\Gamma_0(\mathfrak{n})$:

Definition 2.2. A rigid holomorphic function $f : \Omega \to C$ is said to be a Drinfeld modular form of weight k, type l for $\Gamma_0(\mathfrak{n})$ if

(1) $f|_{k,l}\gamma = f$, $\forall \gamma \in \Gamma_0(\mathfrak{n})$,

(2) f is holomorphic at the cusps of $\Gamma_0(\mathfrak{n})$.

The space of Drinfeld modular forms of weight k, type l for $\Gamma_0(\mathfrak{n})$ is denoted by $M_{k,l}(\Gamma_0(\mathfrak{n}))$. Furthermore, if f vanishes at the cusps of $\Gamma_0(\mathfrak{n})$, then we say f is a Drinfeld cusp form of weight k, type l for $\Gamma_0(\mathfrak{n})$ and the space of such forms is denoted by $S_{k,l}(\Gamma_0(\mathfrak{n}))$.

If $k \not\equiv 2l \pmod{q-1}$, then $M_{k,l}(\Gamma_0(\mathfrak{n})) = \{0\}$. So, without loss of generality, we can assume that $k \equiv 2l \pmod{q-1}$. Let $u(z) := \frac{1}{e_L(\tilde{\pi}z)}$, where $e_L(z) := z\prod_{0\neq\lambda\in L}(1-\frac{z}{\lambda})$ be the exponential function attached to the lattice L. Then, each Drinfeld modular form $f \in M_{k,l}(\Gamma_0(\mathfrak{n}))$ has a unique u-series expansion at ∞ given by $f = \sum_{i=0}^{\infty} a_f(i)u^i$. Since $\left(\begin{smallmatrix} \zeta & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{n})$ for $\zeta \in \mathbb{F}_q^\times$, condition (1) of Definition 2.2 implies $a_f(i) = 0$ if $i \not\equiv l \pmod{q-1}$. Hence, the u-series expansion of f at ∞ can be written as $\sum_{0\leq\, i\equiv l \mod (q-1)} a_f(i)u^i$. Any Drinfeld modular form of type $l \neq 0$ is a cusp form, i. e., $M_{k,l}(\Gamma_0(\mathfrak{n})) = S_{k,l}(\Gamma_0(\mathfrak{n}))$.

2.1. **Examples.** We now give some examples of Drinfeld modular forms.

Example 2.3 ([Gos80], [Gek88])**.** Let $d \in \mathbb{N}$. For $z \in \Omega$, the function

$$g_d(z) := (-1)^{d+1}\tilde{\pi}^{1-q^d} L_d \sum_{\substack{a,b\in\mathbb{F}_q[T] \\ (a,b)\neq(0,0)}} \frac{1}{(az+b)^{q^d-1}}$$

is a Drinfeld modular form of weight q^d-1, type 0 for $\mathrm{GL}_2(A)$, where $\tilde{\pi}$ is the Carlitz period and $L_d := (T^q-T)\dots(T^{q^d}-T)$ is the least common multiple of all monics of degree d. We refer g_d as an Eisenstein series and it does not vanish at ∞.

Example 2.4 ([Gos80a], [Gek88])**.** For $z \in \Omega$, the function

$$\Delta(z) := (T-T^{q^2})\tilde{\pi}^{1-q^2}E_{q^2-1} + (T^q-T)^q\tilde{\pi}^{1-q^2}(E_{q-1})^{q+1},$$

is a Drinfeld cusp form of weight q^2-1, type 0 for $\mathrm{GL}_2(A)$, where $E_k(z) = \sum_{(0,0)\neq(a,b)\in A^2} \frac{1}{(az+b)^k}$.

Example 2.5 ([Gek88])**.** For $z \in \Omega$, the function

$$h(z) := \sum_{\gamma=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in H\backslash \mathrm{GL}_2(A)} \frac{\det\gamma}{(cz+d)^{q+1}}u(\gamma z),$$

where $H = \left\{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(A)\right\}$, is a Drinfeld cusp form of weight $q+1$, type 1 for $\mathrm{GL}_2(A)$.

We end this section by introducing an important function E, which is not modular. In [Gek88], Gekeler defined the function $E(z) := \frac{1}{\tilde{\pi}}\sum_{\substack{a\in\mathbb{F}_q[T] \\ a \text{ monic}}}(\sum_{b\in\mathbb{F}_q[T]}\frac{a}{az+b})$, which is analogous to the Eisenstein series of weight 2 over $\mathbb{Q}$. For any prime ideal $\mathfrak{p} = (P)$, using E, we can construct the following Drinfeld modular form

$$E_P(z) := E(z) - PE(Pz) \in S_{2,1}(\Gamma_0(\mathfrak{p})). \tag{2.2}$$

(cf. [DK21, Proposition 3.3] for a detailed discussion about E_P).

3. Certain important operators

In this section, we recall certain important operators and study their properties.

3.1. **Atkin-Lehner operators:** Let $\mathfrak{r}$, $\mathfrak{n}$ be two ideals of A generated by monic polynomials r, n, respectively, with $\mathfrak{r} \mid \mathfrak{n}$. The following definition can be found in [Sch96, Page 331].

Definition 3.1. For $\mathfrak{r}||\mathfrak{n}$ (i.e., $\mathfrak{r}|\mathfrak{n}$ with $(\mathfrak{r}, \frac{\mathfrak{n}}{\mathfrak{r}}) = 1$), the (partial) Atkin-Lehner operator $W_{\mathfrak{r}}^{(\mathfrak{n})}$ is defined by the action of the matrix $\left(\begin{smallmatrix} ar & b \\ cn & dr \end{smallmatrix}\right)$ on $M_{k,l}(\Gamma_0(\mathfrak{n}))$, where $a,b,c,d \in A$ such that $adr^2 - bcn = \zeta \cdot r$ for some $\zeta \in \mathbb{F}_q^\times$.

By [DK21, Proposition 3.2], the action of $W_{\mathfrak{r}}^{(\mathfrak{n})}$ on $M_{k,l}(\Gamma_0(\mathfrak{n}))$ is well-defined (here the action of the slash operator is different from the one in [DK21]). Assume that $\mathfrak{p}^\alpha||\mathfrak{n}$ with $\alpha \in \mathbb{N}$. We now fix some representatives for the (partial) Atkin-Lehner operators $W_{\mathfrak{p}^\alpha}^{(\mathfrak{n})}$ and $W_{\mathfrak{p}^{\alpha-1}}^{(\mathfrak{m})}$.

Definition 3.2. For $f \in S_{k,l}(\Gamma_0(\mathfrak{n}))$, we write $f|_{k,l}W^{(\mathfrak{n})}_{\mathfrak{p}^\alpha} := f|_{k,l}\left(\begin{smallmatrix} P^\alpha & b \\ n & P^\alpha d\end{smallmatrix}\right)$, where $b, d \in A$ such that $P^{2\alpha}d - nb = P^\alpha$. Since $(P^\alpha, \frac{n}{P^\alpha}) = 1$, such $b, d \in A$ exist.

Write $n = Pm$ and $\mathfrak{m} = (m)$. When $\alpha \geq 2$, we can take $\left(\begin{smallmatrix} P^{\alpha-1} & b \\ m & P^\alpha d\end{smallmatrix}\right)$ as a representative for the (partial) Atkin-Lehner operator $W^{(\mathfrak{m})}_{\mathfrak{p}^{\alpha-1}}$.

Lemma 3.3. *The operator $|_{k,l}W^{(\mathfrak{n})}_{\mathfrak{p}^\alpha}$ on $S_{k,l}(\Gamma_0(\mathfrak{n}))$ defines an endomorphism and for all $f \in S_{k,l}(\Gamma_0(\mathfrak{n}))$, we have $(f|_{k,l}W^{(\mathfrak{n})}_{\mathfrak{p}^\alpha})|_{k,l}W^{(\mathfrak{n})}_{\mathfrak{p}^\alpha} = P^{\alpha(2l-k)}f$*

Proof. Since $W^{(\mathfrak{n})}_{\mathfrak{p}^\alpha} \cdot W^{(\mathfrak{n})}_{\mathfrak{p}^\alpha} = \left(\begin{smallmatrix} P^\alpha & 0 \\ 0 & P^\alpha\end{smallmatrix}\right)\gamma$ for some $\gamma \in \Gamma_0(\mathfrak{n})$, the lemma follows. □

Lemma 3.4. *For $i = 1, 2$, let $\mathfrak{p}_i$ be two distinct prime ideals of A such that $\mathfrak{p}_i^{\alpha_i}||\mathfrak{n}$ for some $\alpha_i \in \mathbb{N}$. Then $W^{(\mathfrak{n})}_{\mathfrak{p}_1^{\alpha_1}}W^{(\mathfrak{n})}_{\mathfrak{p}_2^{\alpha_2}} = W^{(\mathfrak{n})}_{\mathfrak{p}_2^{\alpha_2}}W^{(\mathfrak{n})}_{\mathfrak{p}_1^{\alpha_1}}$.*

Proof. The proof of the lemma follows from $W^{(\mathfrak{n})}_{\mathfrak{p}_1^{\alpha_1}}W^{(\mathfrak{n})}_{\mathfrak{p}_2^{\alpha_2}} = W^{(\mathfrak{n})}_{\mathfrak{p}_1^{\alpha_1}\mathfrak{p}_2^{\alpha_2}} = W^{(\mathfrak{n})}_{\mathfrak{p}_2^{\alpha_2}}W^{(\mathfrak{n})}_{\mathfrak{p}_1^{\alpha_1}}$. □

3.2. **Hecke operators:** We now recall the definitions of $T_\mathfrak{p}$ and $U_\mathfrak{p}$-operators.

Definition 3.5. For $f \in S_{k,l}(\Gamma_0(\mathfrak{n}))$, we define

$$T_\mathfrak{p}(f) := P^{k-l}\sum_{\substack{Q\in A\\ \deg Q<\deg P}} f|_{k,l}\left(\begin{smallmatrix} 1 & Q \\ 0 & P\end{smallmatrix}\right) + P^{k-l}f|_{k,l}\left(\begin{smallmatrix} P & 0 \\ 0 & 1\end{smallmatrix}\right) \quad \text{if } \mathfrak{p} \nmid \mathfrak{n},$$
$$U_\mathfrak{p}(f) := P^{k-l}\sum_{\substack{Q\in A\\ \deg Q<\deg P}} f|_{k,l}\left(\begin{smallmatrix} 1 & Q \\ 0 & P\end{smallmatrix}\right) \quad \text{if } \mathfrak{p} \mid \mathfrak{n}.$$

The commutativity of the $T_\mathfrak{p}$ and $U_\mathfrak{p}$-operators is content of the next proposition.

Proposition 3.6. *Let $\mathfrak{n}$ be an ideal of A and $\mathfrak{p}_1, \mathfrak{p}_2$ be two distinct prime ideals of A generated by monic irreducible polynomials P_1, P_2 respectively. Suppose that $\mathfrak{p}_1 \mid \mathfrak{n}$. Then, $U_{\mathfrak{p}_1}$ commutes with $U_{\mathfrak{p}_2}$ (resp., with $T_{\mathfrak{p}_2}$) if $\mathfrak{p}_2 \mid \mathfrak{n}$ (resp., if $\mathfrak{p}_2 \nmid \mathfrak{n}$) as operators on $S_{k,l}(\Gamma_0(\mathfrak{n}))$.*

Proof. Since P_1 and P_2 are distinct primes, for any $b \in A$ with $\deg b < \deg P_1$ there exists a unique $b' \in A$ with $\deg b' < \deg P_1$ such that $P_1|(b - b'P_2)$. Thus, $\left(\begin{smallmatrix} 1 & \frac{b-b'P_2}{P_1} \\ 0 & 1\end{smallmatrix}\right) \in \Gamma_0(\mathfrak{n})$ and $\left(\begin{smallmatrix} 1 & b \\ 0 & P_1\end{smallmatrix}\right)\left(\begin{smallmatrix} P_2 & 0 \\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & \frac{b-b'P_2}{P_1} \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} P_2 & 0 \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b' \\ 0 & P_1\end{smallmatrix}\right)$. Now the result follows from Definition 3.5 and the following equality $\sum_{\substack{b\in A\\ \deg b<\deg P_1}}\sum_{\substack{d\in A\\ \deg d<\deg P_2}}\left(\begin{smallmatrix} 1 & b+dP_1 \\ 0 & P_1P_2\end{smallmatrix}\right) = \sum_{\substack{c\in A\\ \deg c<\deg P_1+\deg P_2}}\left(\begin{smallmatrix} 1 & c \\ 0 & P_1P_2\end{smallmatrix}\right) = \sum_{\substack{d'\in A\\ \deg d'<\deg P_2}}\sum_{\substack{b'\in A\\ \deg b'<\deg P_1}}\left(\begin{smallmatrix} 1 & d'+b'P_2 \\ 0 & P_1P_2\end{smallmatrix}\right)$. □

3.3. **The Trace operators.** We define the trace operators and mention some of its properties.

Definition 3.7. For any ideal $\mathfrak{r}|\mathfrak{n}$, we define the trace operator $\mathrm{Tr}^\mathfrak{n}_\mathfrak{r} : M_{k,l}(\Gamma_0(\mathfrak{n})) \longrightarrow M_{k,l}(\Gamma_0(\mathfrak{r}))$ by $\mathrm{Tr}^\mathfrak{n}_\mathfrak{r}(f) = \sum_{\gamma\in\Gamma_0(\mathfrak{n})\backslash\Gamma_0(\mathfrak{r})} f|_{k,l}\gamma$.

We conclude this section with a proposition where we explicitly compute the action of the trace operator in terms of the (partial) Atkin-Lehner operators and the Hecke operators.

Proposition 3.8. *Let $\mathfrak{m}, \mathfrak{n}$ be two ideals of A generated by monic polynomials m, n, respectively, such that $n = Pm$. Let $\alpha \in \mathbb{N}$ such that $P^\alpha||n$. If $f \in S_{k,l}(\Gamma_0(\mathfrak{n}))$, then*

$$\mathrm{Tr}^\mathfrak{n}_\mathfrak{m}(f) = \begin{cases} f + P^{-l}U_\mathfrak{p}(f|W^{(\mathfrak{n})}_\mathfrak{p}) & \text{if } \alpha = 1, \\ P^{-l-(\alpha-1)(2l-k)}U_\mathfrak{p}(f|W^{(\mathfrak{n})}_{\mathfrak{p}^\alpha})|_{k,l}W^{(\mathfrak{m})}_{\mathfrak{p}^{\alpha-1}} & \text{if } \alpha \geq 2.\end{cases}$$

Proof. If $\alpha = 1$, then this proposition is exactly [DK21, Proposition 3.6]. When $\mathfrak{n}$ is a prime ideal, this coincides with [Vin14, Proposition 3.8]. Note that, the action of the slash operator here is different from there.

Now, we let $\alpha \geq 2$. By definition, we have

$$\begin{aligned} U_P(f|W^{(\mathfrak{n})}_{\mathfrak{p}^\alpha}) &= P^{k-l} \sum_{\deg Q<\deg P} f|_{k,l}\left(\begin{smallmatrix} P^\alpha & b \\ n & P^\alpha d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & Q \\ 0 & P \end{smallmatrix}\right) = P^{k-l} \sum_{\deg Q<\deg P} f|_{k,l}\left(\begin{smallmatrix} P^\alpha & P^\alpha Q+bP \\ n & nQ+P^{\alpha+1}d \end{smallmatrix}\right) \\ &= P^{k-l} \sum_{\deg Q<\deg P} f|_{k,l}\left(\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right)\left(\begin{smallmatrix} P^{\alpha-1} & P^{\alpha-1}Q+b \\ m & mQ+P^\alpha d \end{smallmatrix}\right) = P^{l} \sum_{\deg Q<\deg P} f|_{k,l}\left(\begin{smallmatrix} P^{\alpha-1} & P^{\alpha-1}Q+b \\ m & mQ+P^\alpha d \end{smallmatrix}\right) \\ &= P^{l} \sum_{\deg Q<\deg P} f|_{k,l}\left(\begin{smallmatrix} 1-mQ & P^{\alpha-1}Q \\ -\frac{m^2}{P^{\alpha-1}}Q & 1+mQ \end{smallmatrix}\right)\left(\begin{smallmatrix} P^{\alpha-1} & b \\ m & P^\alpha d \end{smallmatrix}\right). \end{aligned}$$

We now show that $\left\{\left(\begin{smallmatrix} 1-mQ & P^{\alpha-1}Q \\ -\frac{m^2}{P^{\alpha-1}}Q & 1+mQ \end{smallmatrix}\right) : \deg Q < \deg P\right\}$ is a set of representatives for $\Gamma_0(\mathfrak{n})\backslash\Gamma_0(\mathfrak{m})$. Let $\left(\begin{smallmatrix} s & t \\ mx & y \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{m})$, where $s,t,x,y \in A$ satisfy $sy - tmx = \zeta \in \mathbb{F}_q^\times$. Let $-\zeta^{-1}sx \equiv Q_1 \pmod P$, where $Q_1 \in A$ such that $\deg Q_1 < \deg P$. Since $P^{\alpha-1}||m$, there exists an unique $Q_2 \in A$ with $\deg Q_2 < \deg P$ such that $\frac{m}{P^{\alpha-1}}Q_2 \equiv 1 \pmod P$. Since $P|m$, the choice of Q_1 and $sy - tmx = \zeta \in \mathbb{F}_q^\times$ implies that $x + yQ_1 \equiv 0 \pmod P$. Let $Q \in A$ with $\deg Q < \deg P$ such that $Q_1Q_2 \equiv Q \pmod P$. Then $x + \frac{m}{P^{\alpha-1}}Qy \equiv 0 \pmod P$. Hence, we get $\left(\begin{smallmatrix} s(1+mQ)+t\frac{m^2}{P^{\alpha-1}}Q & t(1-mQ)-sP^{\alpha-1}Q \\ mx(1+mQ)+y\frac{m^2}{P^{\alpha-1}}Q & y(1-mQ)-mxP^{\alpha-1}Q \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{n})$ and

$$\left(\begin{smallmatrix} s & t \\ mx & y \end{smallmatrix}\right) = \left(\begin{smallmatrix} s(1+mQ)+t\frac{m^2}{P^{\alpha-1}}Q & t(1-mQ)-sP^{\alpha-1}Q \\ mx(1+mQ)+y\frac{m^2}{P^{\alpha-1}}Q & y(1-mQ)-mxP^{\alpha-1}Q \end{smallmatrix}\right)\left(\begin{smallmatrix} 1-mQ & P^{\alpha-1}Q \\ -\frac{m^2}{P^{\alpha-1}}Q & 1+mQ \end{smallmatrix}\right).$$

Thus, the set $\left\{\left(\begin{smallmatrix} 1-mQ & P^{\alpha-1}Q \\ -\frac{m^2}{P^{\alpha-1}}Q & 1+mQ \end{smallmatrix}\right) : \deg Q < \deg P\right\}$ forms a complete set of representatives for $\Gamma_0(\mathfrak{n})\backslash\Gamma_0(\mathfrak{m})$. Therefore

$$U_P(f|W^{(\mathfrak{n})}_{\mathfrak{p}^\alpha}) = P^{l} \sum_{\deg Q<\deg P} f|_{k,l}\left(\begin{smallmatrix} 1-mQ & P^{\alpha-1}Q \\ -\frac{m^2}{P^{\alpha-1}}Q & 1+mQ \end{smallmatrix}\right)\left(\begin{smallmatrix} P^{\alpha-1} & b \\ m & P^\alpha d \end{smallmatrix}\right) = P^l(\mathrm{Tr}^{\mathfrak{n}}_{\mathfrak{m}}f)|_{k,l}W^{(\mathfrak{m})}_{\mathfrak{p}^{\alpha-1}}.$$

Applying $W^{(\mathfrak{m})}_{\mathfrak{p}^{\alpha-1}}$ operator on both sides, the proposition follows from Lemma 3.3. $\square$

Corollary 3.9. *Let $\mathfrak{p}, \mathfrak{m}$ be with $(\mathfrak{p}, \mathfrak{m}) = 1$. If $f \in S_{k,l}(\Gamma_0(\mathfrak{p}))$, then $\mathrm{Tr}^{\mathfrak{pm}}_{\mathfrak{m}}(f) = \mathrm{Tr}^{\mathfrak{p}}_{1}(f)$.*

Proof. Since $f|W^{(\mathfrak{pm})}_{\mathfrak{p}} = f|W^{(\mathfrak{p})}_{\mathfrak{p}}$, the result follows from Proposition 3.8. $\square$

4. $\mathfrak{p}$-oldforms and $\mathfrak{p}$-newforms for level $\mathfrak{pm}$

Let $\mathfrak{p}$ be a prime ideal of A. Throughout this section, we consider $\mathfrak{m}$ an ideal of A generated by a monic polynomial m such that $\mathfrak{p} \nmid \mathfrak{m}$. We first recall the definitions of $\mathfrak{p}$-oldforms and $\mathfrak{p}$-newforms (cf. [BV20], [Val22]). Consider the map

$$(\delta_1, \delta_P) : (S_{k,l}(\Gamma_0(\mathfrak{m})))^2 \longrightarrow S_{k,l}(\Gamma_0(\mathfrak{pm})) \text{ defined by } (f, g) \longrightarrow \delta_1 f + \delta_P g,$$

where δ_1, $\delta_P : S_{k,l}(\Gamma_0(\mathfrak{m})) \to S_{k,l}(\Gamma_0(\mathfrak{pm}))$ given by $\delta_1(f) = f$ and $\delta_P(f) = f|_{k,l}\left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Definition 4.1. The space of $\mathfrak{p}$-oldforms $S^{\mathfrak{p}-\mathrm{old}}_{k,l}(\Gamma_0(\mathfrak{pm}))$ of level $\mathfrak{pm}$ is defined as the subspace of $S_{k,l}(\Gamma_0(\mathfrak{pm}))$ generated by the image of (δ_1, δ_P).

Definition 4.2. The space of $\mathfrak{p}$-newforms $S^{\mathfrak{p}-\mathrm{new}}_{k,l}(\Gamma_0(\mathfrak{pm}))$ of level $\mathfrak{pm}$ is defined as

$$S^{\mathfrak{p}-\mathrm{new}}_{k,l}(\Gamma_0(\mathfrak{pm})) := \mathrm{Ker}(\mathrm{Tr}^{\mathfrak{pm}}_{\mathfrak{m}}) \cap \mathrm{Ker}(\mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}}), \quad \text{where } \mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}} f := \mathrm{Tr}^{\mathfrak{pm}}_{\mathfrak{m}}(f|W^{(\mathfrak{pm})}_{\mathfrak{p}}).$$

We wish to understand Conjecture 1.1 for prime $\mathfrak{p}$, higher levels $\mathfrak{pm}$. We now formulate it as a question and provide some evidences in favor of it. More precisely:

Question 4.3 (For level $\mathfrak{pm}$)**.** Suppose $\mathfrak{m}$ is an ideal of A such that $\mathfrak{p} \nmid \mathfrak{m}$.

(1) $\ker(T_{\mathfrak{p}}) = 0$, where $T_{\mathfrak{p}} \in \mathrm{End}(S_{k,l}(\Gamma_0(\mathfrak{m})))$,
(2)
$$S_{k,l}(\Gamma_0(\mathfrak{pm})) = S^{\mathfrak{p}-\mathrm{old}}_{k,l}(\Gamma_0(\mathfrak{pm})) \oplus S^{\mathfrak{p}-\mathrm{new}}_{k,l}(\Gamma_0(\mathfrak{pm})), \tag{4.1}$$
(3) The $U_{\mathfrak{p}}$-operator is diagonalizable on $S_{k,l}(\Gamma_0(\mathfrak{pm}))$.

When we say that "Question 4.3 is true for level $\mathfrak{pm}$", we mean all the statements of Question 4.3 are true. We first show, if $\mathfrak{m} = A, \mathfrak{p} = (P)$ with $\deg P = 1$, then Question 4.3 is true for level $\mathfrak{p}$ if dim $S_{k,l}(\mathrm{GL}_2(A)) \leq 1$. In particular, Conjecture 1.1 is true for $S_{k,l}(\Gamma_0(T))$. Furthermore, we show the direct sum decomposition in Question 4.3(2) holds for $S_{k,l}(\Gamma_0(\mathfrak{p}))$ if $\dim S_{k,l}(\mathrm{GL}_2(A)) \leq 2$. Finally, we give some evidences in the support of Question 4.3 for level $\mathfrak{pm}$.

4.1. **Question 4.3 when dim $S_{k,l}(\mathrm{GL}_2(A)) \leq 2$:** We now discuss on implications of Question 4.3(1),(2) to Question 4.3(3). If Question 4.3(2) is true, then the diagonalizability of the $U_\mathfrak{p}$-operator on $S_{k,l}(\Gamma_0(\mathfrak{pm}))$ depends on that of the $U_\mathfrak{p}$-operators on $S_{k,l}^{\mathfrak{p}-\mathrm{old}}(\Gamma_0(\mathfrak{pm}))$, $S_{k,l}^{\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{pm}))$. By [BV20, Remark 2.17], the $U_\mathfrak{p}$-operator is diagonalizable on $S_{k,l}^{\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{pm}))$. However, the $U_\mathfrak{p}$-operator is diagonalizable on $S_{k,l}^{\mathfrak{p}-\mathrm{old}}(\Gamma_0(\mathfrak{pm}))$ if and only if the $T_\mathfrak{p}$-operator is diagonalizable on $S_{k,l}(\Gamma_0(\mathfrak{m}))$ and is injective (cf. [BV20, Remark 2.4]). Therefore, if Question 4.3(1),(2) are true, then Question 4.3(3) is equivalent to check the diagonalizability of the $T_\mathfrak{p}$-operator on $S_{k,l}(\Gamma_0(\mathfrak{m}))$.

4.1.1. *Reformulation of Question 4.3(2).* In [Val22], Valentino gave a necessary and sufficient condition for Question 4.3(2) to hold. More precisely:

Theorem 4.4. [Val22, Theorem 3.15] *The map* $\mathrm{Id} - P^{k-2l}(\mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}})^2$ *is bijective on* $S_{k,l}(\Gamma_0(\mathfrak{pm}))$ *if and only if Question 4.3(2) holds.*

We now rephrase Theorem 4.4 in terms of the eigenvalues of the $T_\mathfrak{p}$-operator.

Proposition 4.5. *The $T_\mathfrak{p}$-operator has no eigenform on $S_{k,l}(\Gamma_0(\mathfrak{m}))$ with eigenvalues $\pm P^{\frac{k}{2}}$ if and only if Question 4.3(2) holds.*

The proof of Proposition 4.5 depends on the following observations. For any $\varphi \in S_{k,l}(\Gamma_0(\mathfrak{m}))$, we have:

$$\begin{aligned} \varphi|_{k,l} W_\mathfrak{p}^{(\mathfrak{pm})} &= \varphi|_{k,l}\left(\begin{smallmatrix} 1 & b \\ m & dP \end{smallmatrix}\right)\left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right) = \varphi|_{k,l}\left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right) = \delta_P \varphi, \\ (\delta_P\varphi)|_{k,l} W_\mathfrak{p}^{(\mathfrak{pm})} &= \varphi|_{k,l}\left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} P & b \\ Pm & dP \end{smallmatrix}\right) = P^{2l-k}\varphi. \end{aligned} \tag{4.2}$$

Combining Proposition 3.8 with (4.2) we obtain

$$\mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}}(\delta_1(\varphi)) = \varphi|_{k,l} W_\mathfrak{p}^{(\mathfrak{pm})} + P^{l-k}(U_\mathfrak{p}(\delta_1(\varphi))) = \delta_P\varphi + P^{l-k}(U_\mathfrak{p}(\delta_1(\varphi))) = P^{l-k}T_\mathfrak{p}\varphi, \tag{4.3}$$

where $W_\mathfrak{p}^{(\mathfrak{pm})} := \left(\begin{smallmatrix} P & b \\ Pm & dP \end{smallmatrix}\right)$ for some $b, d \in A$ with $dP^2 - bPm = P$.

Proof of Proposition 4.5. The proof of Theorem 4.4 implies that if $f \in \ker(\mathrm{Id} - P^{k-2l}(\mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}})^2)$, then $f \in \mathrm{Im}(\delta_1)$. Thus $\ker(\mathrm{Id} - P^{k-2l}(\mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}})^2) \subseteq S_{k,l}(\Gamma_0(\mathfrak{m}))$. Therefore, $\mathrm{Id} - P^{k-2l}(\mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}})^2$ is bijective on $S_{k,l}(\Gamma_0(\mathfrak{pm}))$ if and only if it is bijective on $S_{k,l}(\Gamma_0(\mathfrak{m}))$.

For any $f \in S_{k,l}(\Gamma_0(\mathfrak{m}))$, (4.3) implies $\mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}}(\mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}}(f)) = P^{l-k}\mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}}(T_\mathfrak{p}(f)) = P^{2l-2k}T_\mathfrak{p}(T_\mathfrak{p}(f))$. Thus, on $S_{k,l}(\Gamma_0(\mathfrak{m}))$, we have $\mathrm{Id} - P^{k-2l}(\mathrm{Tr'}^{\mathfrak{pm}}_{\mathfrak{m}})^2 = \mathrm{Id} - P^{-k}T_\mathfrak{p}^2$. Observe that the map $\mathrm{Id} - P^{-k}T_\mathfrak{p}^2$ is bijective on $S_{k,l}(\Gamma_0(\mathfrak{m}))$ if and only if the $T_\mathfrak{p}$-operator has no eigenform on $S_{k,l}(\Gamma_0(\mathfrak{m}))$ with eigenvalues $\pm P^{\frac{k}{2}}$. Now the proposition follows from Theorem 4.4. $\square$

We now prove a part of the Conjecture 1.1 for $S_{k,l}(\Gamma_0(T))$ when dim $S_{k,l}(\mathrm{GL}_2(A)) \leq 2$. We first prove that, Conjecture 1.1 is true for $S_{k,l}(\Gamma_0(T))$ when dim $S_{k,l}(\mathrm{GL}_2(A)) \leq 1$.

Theorem 4.6. *For $\mathfrak{m} = A, \deg P = 1$, Question 4.3 is true for $S_{k,l}(\Gamma_0(P))$ when $\dim S_{k,l}(\mathrm{GL}_2(A)) \leq 1$. In particular, Conjecture 1.1 is true for $S_{k,l}(\Gamma_0(T))$ when $\dim S_{k,l}(\mathrm{GL}_2(A)) \leq 1$.*

Theorem 4.6 can be thought of as a continuation of the work done by Bandini and Valentino in [BV19a], [BV20], and [BV22]. Their method mainly involves harmonic cocycles, the trace maps Tr and Tr$'$, and the linear algebra interpretation of the Hecke operators $T_\mathfrak{p}$ and $U_\mathfrak{p}$-operators. However, we prove Theorem 4.6 based on the analysis of the Fourier coefficients of the image of an element via the $T_\mathfrak{p}$-operator, which may be suitable for generalizations. Finally, recall that $\dim M_{k,l}(\mathrm{GL}_2(A)) = \left[\frac{k-l(q+1)}{q^2-1}\right] + 1$ (cf. [Cor97, Proposition 4.3]). By [Gek88, Theorem 5.13], the graded algebra $\oplus_{k,l} M_{k,l}(\mathrm{GL}_2(A))$ is generated by g_1, h .

Proof of Theorem 4.6. Suppose $\dim S_{k,l}(\mathrm{GL}_2(A)) = 0$. Then Question 4.3(1) is trivially true. Question 4.3(2) and (3) are true by Proposition 4.5, by the diagonalizability of the $T_\mathfrak{p}$-operator on $S_{k,l}(\mathrm{GL}_2(A))$.

Now, suppose $\dim S_{k,l}(\mathrm{GL}_2(A)) = 1$. Clearly, the $T_\mathfrak{p}$-operator is diagonalizable on $S_{k,l}(\mathrm{GL}_2(A))$. Therefore, combining Proposition 4.5 with the discussions in §4.1, Question 4.3 for the level (P) is true if we show $\ker(T_\mathfrak{p}) = 0$ and the $T_\mathfrak{p}$-operator has no eigenform on $S_{k,l}(\mathrm{GL}_2(A))$ with eigenvalues $\pm P^{\frac{k}{2}}$. We prove this statement in two cases, i.e., for $l \neq 0$ and $l = 0$.

We first consider the case $l \neq 0$. In this case, $S_{k,l}(\mathrm{GL}_2(A)) = \langle g_1^x h^l \rangle$ for some $x \in \{0, \ldots, q\}$ such that $k = x(q-1) + l(q+1)$. The u-series expansions of g_1, h are given by

$$g_1 = 1 - (T^q - T)u^{q-1} - (T^q - T)u^{(q-1)(q^2-q+1)} + \cdots \in A[[u]],$$

$$h = -u - u^{1+(q-1)^2} + (T^q - T)u^{1+q(q-1)} - u^{1+(2q-2)(q-1)} + \cdots \in A[[u]].$$

Therefore, $g_1^x h^l = (-1)^l \sum_{i=0}^{x} (-1)^i \binom{x}{i} (T^q - T)^i u^{i(q-1)+l} + O(u^{(q-1)^2+l}) \in A[[u]]$. Let $T_\mathfrak{p}(g_1^x h^l) = \sum_{j=0}^{\infty} a_{T_\mathfrak{p}(g_1^x h^l)}(j(q-1)+l) u^{j(q-1)+l}$. By [Gek88, Example 7.4], we have

$$a_{T_\mathfrak{p}(g_1^x h^l)}(l) = \sum_{0 \le j < l} \binom{l-1}{j} P^{l-j} a_{g_1^x h^l}(j(q-1)+l) \in A.$$

We define $x_0 := \min\{x, l-1\}$. Then, $a_{T_\mathfrak{p}(g_1^x h^l)}(l) = \sum_{0 \le j \le x_0} \binom{l-1}{j} P^{l-j} (-1)^{l+j} \binom{x}{j} (T^q - T)^j$. Clearly, the set $\{0 \le j \le x_0 \mid \binom{l-1}{j}\binom{x}{j} \neq 0\}$ is non-empty and let $j_{\max}$ be its maximum. Since $\deg(P^{l-j}(T^q - T)^j) < \deg(P^{l-(j+1)}(T^q - T)^{j+1})$, we get

$$0 < \deg(a_{T_\mathfrak{p}(g_1^x h^l)}(l)) = l + j_{\max}(q-1) \le l + x_0(q-1) < \frac{x(q-1) + l(q+1)}{2}. \tag{4.4}$$

The first inequality in (4.4) shows that $\ker(T_\mathfrak{p}) = 0$. The inequality $\deg(a_{T_\mathfrak{p}(g_1^x h^l)}(l)) < \frac{x(q-1)+l(q+1)}{2}$ in (4.4) shows that $T_\mathfrak{p}(g_1^x h^l)$ cannot be equal to $\pm P^{\frac{x(q-1)+l(q+1)}{2}} g_1^x h^l$. In particular, the $T_\mathfrak{p}$-operator has no eigenform on $S_{k,l}(\mathrm{GL}_2(A))$ with eigenvalues $\pm P^{\frac{k}{2}}$.

We now consider the case of $l = 0$. In this case, $S_{k,0}(\mathrm{GL}_2(A)) = \langle g_1^x \Delta \rangle$ for some $x \in \{0, \ldots, q\}$ such that $k = x(q-1) + (q^2-1)$. Recall that $\Delta = -u^{q-1} + u^{q(q-1)} - (T^q - T)u^{(q+1)(q-1)} + O(u^{(q^2-q+1)(q-1)}) \in A[[u]]$. Hence

$$g_1^x \Delta = \sum_{i=0}^{x} \binom{x}{i} (-1)^{i+1} (T^q - T)^i u^{(i+1)(q-1)} + O(u^{q(q-1)}) \in A[[u]].$$

In this case, we consider the coefficient $a_{T_\mathfrak{p}(g_1^x \Delta)}(q-1)$ to prove our claims. Since $a_{g_1^x\Delta}(0) = 0$, we have $a_{T_\mathfrak{p}(g_1^x \Delta)}(q-1) = \sum_{0 \le j < q-1} \binom{q-2}{j} P^{q-1-j} a_{g_1^x \Delta}((j+1)(q-1))$ (cf. [Gek88, Example 7.4]). We define $y_0 := \min\{x, q-2\}$. Then,

$$a_{T_\mathfrak{p}(g_1^x \Delta)}(q-1) = \sum_{0 \le j \le y_0} \binom{q-2}{j} P^{q-1-j} (-1)^{j+1} \binom{x}{j} (T^q - T)^j.$$

Arguing as in the previous case, i.e., $l \neq 0$, we get $0 < \deg(a_{T_\mathfrak{p}(g_1^x \Delta)}(q-1)) < \frac{x(q-1)+(q^2-1)}{2}$, which shows that $\ker(T_\mathfrak{p}) = 0$ and the $T_\mathfrak{p}$-operator has no eigenform on $S_{k,l}(\mathrm{GL}_2(A))$ with eigenvalues $\pm P^{\frac{k}{2}}$. This completes the proof of the proposition. □

We now show that, a part of the Conjecture 1.1 is true for $S_{k,l}(\Gamma_0(T))$ when dim $S_{k,l}(\mathrm{GL}_2(A)) = 2$. More precisely,

Theorem 4.7. *Let $\mathfrak{m} = A$ and $\deg P = 1$. If $\dim S_{k,l}(\mathrm{GL}_2(A)) = 2$, then the direct sum decomposition in Question 4.3(2) is true for $S_{k,l}(\Gamma_0(\mathfrak{p}))$.*

Proof. By Proposition 4.5, it is enough to show $T_\mathfrak{p}$-operator has no eigenform on $S_{k,l}(\mathrm{GL}_2(A))$ with eigenvalues $\pm P^{k/2}$. We give a complete proof only for $l \neq 0$ and the proof is similar when $l = 0$. So, we assume $l \neq 0$.

Since $\dim S_{k,l}(\mathrm{GL}_2(A)) = 2$, $S_{k,l}(\mathrm{GL}_2(A)) = \langle g_1^y h^l, g_1^x \Delta h^l \rangle$ for some $y \in \{q+1, \ldots, 2q+1\}$ such that $k = y(q-1) + l(q+1)$ and where $x := y - (q+1)$. There are two cases to be considered.

We first assume that $(y,l) \neq (2q+1,1)$: Recall the following u-expansions

$$g_1^y = \begin{cases} \sum_{i=0}^{y} \binom{y}{i}(-1)^i (T^q - T)^i u^{i(q-1)} + O(u^{(l+q)(q-1)}) & \text{if } y < l + (q-1), \\ \sum_{i=0}^{l+(q-1)} \binom{y}{i}(-1)^i (T^q - T)^i u^{i(q-1)} + O(u^{(l+q)(q-1)}) & \text{if } y \geq l + (q-1). \end{cases}$$

$$g_1^x = \sum_{i=0}^{x} \binom{x}{i}(-1)^i (T^q - T)^i u^{i(q-1)} + O(u^{(q-1)(q^2-q+1)}). \tag{4.5}$$

$$h^l = (-1)^l u^l + (-1)^l l u^{(q-1)^2+l} + (-1)^{l-1} l (T^q - T) u^{q(q-1)+l} + O(u^{(l+q)(q-1)+l}). \tag{4.6}$$

$$\Delta = -u^{q-1} + u^{q(q-1)} - (T^q - T) u^{(q+1)(q-1)} + O(u^{(q^2-q+1)(q-1)}).$$

$$\Delta h^l = (-1)^{l+1} u^{q-1+l} + (-1)^l (1-l) u^{q(q-1)+l} + (-1)^l (l-1)(T^q - T) u^{(q^2-1)+l} + O(u^{(l+q)(q-1)+l}) \tag{4.7}$$

Finally, we have the required u-expansions of $g_1^x \Delta h^l$ and $g_1^y h^l$ as

$$g_1^x \Delta h^l = \begin{cases} (-1)^{l+1} \sum_{i=1}^{x+1} \binom{x}{i-1}(-1)^{i-1} (T^q - T)^{i-1} u^{i(q-1)+l} + O(u^{l(q-1)+l}) & \text{if } x+1 < l, \\ (-1)^{l+1} \sum_{i=1}^{l-1} \binom{x}{i-1}(-1)^{i-1} (T^q - T)^{i-1} u^{i(q-1)+l} + O(u^{l(q-1)+l}) & \text{if } x+1 \geq l. \end{cases}$$

$$g_1^y h^l = (-1)^l \sum_{i=0}^{l-1} \binom{y}{i}(-1)^i (T^q - T)^i u^{i(q-1)+l} + O(u^{l(q-1)+l}).$$

We first show that $T_{\mathfrak{p}}(g_1^x \Delta h^l) \neq \pm P^{\frac{x(q-1)+(q^2-1)+l(q+1)}{2}} g_1^x \Delta h^l$. The $(l+(q-1))$-th coefficient of $T_{\mathfrak{p}}(g_1^x \Delta h^l)$ is given by (cf. [Gek88, Example 7.4])

$$a_{T_{\mathfrak{p}}(g_1^x \Delta h^l)}(l+(q-1)) = \sum_{0 \leq i < l+q-1} \binom{l+q-2}{i} P^{l+q-1-i} a_{g_1^x \Delta h^l}(l+(i+1)(q-1)). \tag{4.8}$$

For $f \in A, g \in A \setminus \{0\}$, $|\frac{f}{g}| := q^{\deg(f)-\deg(g)}$. Now, take the norm of $a_{T_{\mathfrak{p}}(g_1^x \Delta h^l)}(l+(q-1))$ to get

$$\begin{aligned} |a_{T_{\mathfrak{p}}(g_1^x \Delta h^l)}(l+(q-1))| &\leq \max_{1 \leq i \leq l+q-1} \{|P^{l+q-i} a_{g_1^x \Delta h^l}(l+i(q-1))|\} \\ &= \max_{1 \leq i \leq l+q-1} \{|P^{l+q-i} \sum_{\substack{\alpha \in \mathbb{N} \cup \{0\}, \beta \in \mathbb{N} \\ \alpha+\beta=i}} a_{g_1^x}(\alpha(q-1)) \cdot a_{\Delta h^l}(\beta(q-1)+l)|\} \end{aligned}$$

By (4.5) and (4.7), we have $a_{g_1^x}(i(q-1)) = 0$ for $x < i \leq l+q-1$ and $a_{\Delta h^l}(\beta(q-1)+l) = 0$ for $1 \leq \beta \leq l+q-1$ with $\beta \notin \{1, q, q+1\}$. Therefore, we get

$$\begin{aligned} |a_{T_{\mathfrak{p}}(g_1^x \Delta h^l)}(l+(q-1))| &\leq \max_{\beta \in \{1,q,q+1\}} \left\{ \max_{\substack{1 \leq i \leq l+q-1, \\ 0 \leq i-\beta \leq x}} \{|P^{l+q-i} a_{g_1^x}((i-\beta)(q-1)) a_{\Delta h^l}(\beta(q-1)+l)|\} \right\} \\ &= \max \left\{ \max_{\substack{1 \leq i \leq l+q-1 \\ 0 \leq i-1 \leq x}} \{q^{i(q-1)+l}\}, \max_{\substack{1 \leq i \leq l+q-1 \\ 0 \leq i-q \leq x}} \{q^{(i-q)(q-1)+l}\}, \max_{\substack{1 \leq i \leq l+q-1 \\ 0 \leq i-q-1 \leq x}} \{q^{(i-q)(q-1)+l}\} \right\} \\ &= \max \left\{ q^{(x+1)(q-1)+l}, q^{x(q-1)+l}, q^{(x+1)(q-1)+l} \right\} = q^{(x+1)(q-1)+l}. \end{aligned}$$

Hence, we have

$$|a_{T_{\mathfrak{p}}(g_1^x \Delta h^l)}(l+(q-1))| \leq q^{(x+1)(q-1)+l}. \tag{4.9}$$

On the other hand, the assumption $(y,l) \neq (2q+1,1)$ implies $(x+1)(q-1)+l < \frac{x(q-1)+(q^2-1)+l(q+1)}{2}$. Since $a_{g_1^x \Delta h^l}(l+(q-1)) = (-1)^{l+1}$, combining the last inequality with (4.9), we get $T_{\mathfrak{p}}(g_1^x \Delta h^l) \neq$

$\pm P^{\frac{x(q-1)+(q^2-1)+l(q+1)}{2}} g_1^x \Delta h^l$. Now, by the same technique, we give an upper bound on the coefficient $a_{T_\mathfrak{p}(g_1^y h^l)}(l+(q-1))$. Recall that,

$$a_{T_\mathfrak{p}(g_1^y h^l)}(l+(q-1)) = \sum_{0 \le i < l+(q-1)} \binom{l+q-2}{i} P^{l+q-1-i} a_{g_1^y h^l}(l+(i+1)(q-1)). \tag{4.10}$$

Now, take the norm of $a_{T_\mathfrak{p}(g_1^y h^l)}(l+(q-1))$ to get

$$\begin{aligned} |a_{T_\mathfrak{p}(g_1^y h^l)}(l+(q-1))| &\le \max_{1 \le i \le l+q-1} \{|P^{l+q-i} a_{g_1^y h^l}(l+i(q-1))|\} \\ &= \max_{1 \le i \le l+q-1} \{|P^{l+q-i} \sum_{\substack{\alpha,\beta \in \mathbb{N} \cup \{0\} \\ \alpha+\beta=i}} a_{g_1^y}(\alpha(q-1)) \cdot a_{h^l}(\beta(q-1)+l)|\}. \end{aligned}$$

By (4.6), we get that $a_{h^l}(\beta(q-1)+l) = 0$ for $0 \le \beta \le l+q-1$ with $\beta \notin \{0, q-1, q\}$.

- When $y < l+q-1$. In this case, $a_{g_1^y}(i(q-1)) = 0$ for $y < i \le l+q-1$. Hence,

$$\begin{aligned} |a_{T_\mathfrak{p}(g_1^y h^l)}(l+(q-1))| &\le \max_{\beta \in \{0,q-1,q\}} \Big\{ \max_{\substack{\beta \in \{0,q-1,q\} \\ 1 \le i \le l+q-1 \\ 0 \le i-\beta \le y}} \{|P^{l+q-i} a_{g_1^y}((i-\beta)(q-1)) a_{h^l}(\beta(q-1)+l)|\} \Big\} \\ &= \max \Big\{ \max_{\substack{1 \le i \le l+q-1 \\ 0 \le i \le y}} \{q^{i(q-1)+l+q}\}, \max_{\substack{1 \le i \le l+q-1 \\ 0 \le i-q+1 \le y}} \{q^{(i-q)(q-1)+l+q}\}, \max_{\substack{1 \le i \le l+q-1 \\ 0 \le i-q \le y}} \{q^{(i-q)(q-1)+l+q}\} \Big\} \\ &= \max \Big\{ q^{y(q-1)+l+q}, q^{(y-1)(q-1)+l+q}, q^{y(q-1)+l+q} \Big\} = q^{y(q-1)+l+q}. \end{aligned}$$

- When $y \ge l+(q-1)$, a similar argument as above gives $|a_{T_\mathfrak{p}(g_1^y h^l)}(l+q-1)| \le q^{(l+q-1)(q-1)+l+q}$.

Finally, we get

$$|a_{T_\mathfrak{p}(g_1^y h^l)}(l+(q-1))| \le \begin{cases} q^{y(q-1)+l+q} & \text{if } y < l+(q-1), \\ q^{(l+(q-1))(q-1)+l+q} & \text{if } y \ge l+(q-1). \end{cases} \tag{4.11}$$

Since $T_\mathfrak{p}(g_1^x \Delta h^l) \ne \pm P^{\frac{x(q-1)+(q^2-1)+l(q+1)}{2}} g_1^x \Delta h^l$, it is now enough to show that there does not exist any $c \in C$ such that

$$T_\mathfrak{p}(g_1^y h^l + c g_1^x \Delta h^l) = \pm P^{\frac{y(q-1)+l(q+1)}{2}} (g_1^y h^l + c g_1^x \Delta h^l) \tag{4.12}$$

holds. On the contrary, suppose there is an element $c \in C$ such that (4.12) holds with " + " sign. A similar argument works with " − " sign as well. The l-th coefficients of $T_\mathfrak{p}(g_1^y h^l)$ and $T_\mathfrak{p}(g_1^x \Delta h^l)$ are given by (cf. [Gek88, Example 7.4])

$$a_{T_\mathfrak{p}(g_1^y h^l)}(l) = (-1)^l \sum_{0 \le j < l} \binom{l-1}{j} P^{l-j} \binom{y}{j} (-1)^j (T^q - T)^j,$$

$$a_{T_\mathfrak{p}(g_1^x \Delta h^l)}(l) = \begin{cases} (-1)^{l+1} \sum_{j=1}^{x+1} \binom{l-1}{j} P^{l-j} \binom{x}{j-1} (-1)^{j-1} (T^q - T)^{j-1} & \text{if } x+1 < l, \\ (-1)^{l+1} \sum_{j=1}^{l-1} \binom{l-1}{j} P^{l-j} \binom{x}{j-1} (-1)^{j-1} (T^q - T)^{j-1} & \text{if } x+1 \ge l. \end{cases}$$

Comparing the l-th coefficients on both sides of (4.12), we get

$$\sum_{0 \le j < l} \binom{l-1}{j} P^{l-j} \binom{y}{j} (-1)^j (T^q - T)^j - c \sum_{j=1}^{x_0} \binom{l-1}{j} P^{l-j} \binom{x}{j-1} (-1)^{j-1} (T^q - T)^{j-1} = P^{\frac{y(q-1)+l(q+1)}{2}}, \tag{4.13}$$

where $x_0 := \min\{x, l-2\} + 1$. If $c \sum_{j=1}^{x_0} \binom{l-1}{j} P^{l-j} \binom{x}{j-1} (-1)^{j-1} (T^q - T)^{j-1} = 0$, then the inequality $lq < \frac{y(q-1)+l(q+1)}{2}$ would imply that both sides of (4.13) have different degrees. So, the term $c \sum_{j=1}^{x_0} \binom{l-1}{j} P^{l-j} \binom{x}{j-1} (-1)^{j-1} (T^q - T)^{j-1} \ne 0$. Let $j_{\max} := \max\{1 \le j \le x_0 | \binom{l-1}{j} \binom{x}{j-1} \ne 0\}$. Then, $|\sum_{j=1}^{x_0} \binom{l-1}{j} P^{l-j} \binom{x}{j-1} (-1)^{j-1} (T^q - T)^{j-1}| = q^{j_{\max}(q-1)+l-q}$. Since $lq < \frac{y(q-1)+l(q+1)}{2}$, it follows that

$$|P^{\frac{y(q-1)+l(q+1)}{2}} - \sum_{0 \le j < l} \binom{l-1}{j} P^{l-j} \binom{y}{j} (-1)^j (T^q - T)^j| = q^{\frac{y(q-1)+l(q+1)}{2}}.$$

Therefore, (4.13) gives us

$$c=-\frac{P^{\frac{y(q-1)+l(q+1)}{2}}-\sum_{0\le j<l}\binom{l-1}{j}P^{l-j}\binom{y}{j}(-1)^j(T^q-T)^j}{\sum_{j=1}^{j_{\max}}\binom{l-1}{j}P^{l-j}\binom{x}{j-1}(-1)^{j-1}(T^q-T)^{j-1}}\in K, \tag{4.14}$$

hence $|c|=q^{\frac{y(q-1)+l(q+1)}{2}-(j_{\max}(q-1)+l-q)}$. Note that $a_{(g_1^y h^l+cg_1^x\Delta h^l)}(l+(q-1))=(-1)^{l+1}y(T^q-T)+(-1)^{l+1}c$. Using the inequality $lq<\frac{y(q-1)+l(q+1)}{2}$, from (4.14) we obtain

$$|a_{(g_1^y h^l+cg_1^x\Delta h^l)}((q-1)+l)|=q^{\frac{y(q-1)+l(q+1)}{2}-(j_{\max}(q-1)+l-q)}. \tag{4.15}$$

Comparing $(q-1)+l$-th coefficients on both sides of (4.12) we get

$$|a_{T_{\mathfrak{p}}(g_1^y h^l+cg_1^x\Delta h^l)}((q-1)+l)|=q^{y(q-1)+l(q+1)-(j_{\max}(q-1)+l-q)}. \tag{4.16}$$

On the other hand, from (4.11) we have

$$\begin{aligned}|a_{T_{\mathfrak{p}}(g_1^y h^l+cg_1^x\Delta h^l)}((q-1)+l)|&\le\max\{|a_{T_{\mathfrak{p}}(g_1^y h^l)}((q-1)+l)|,|c|\cdot|a_{T_{\mathfrak{p}}(g_1^x\Delta h^l)}((q-1)+l)|\}\\&\le\max\{q^{y_0(q-1)+l+q},q^{\frac{y(q-1)+l(q+1)}{2}-(j_{\max}(q-1)+l-q)}\cdot q^{(x+1)(q-1)+l}\}\end{aligned}$$

where $y_0:=\min\{y,l+(q-1)\}$. Since $0\le j_{\max}<l$, an easy verification shows that $q^{y_0(q-1)+l+q}<q^{y(q-1)+l(q+1)-(j_{\max}(q-1)+l-q)}$. Moreover, the inequality $(x+1)(q-1)+l<\frac{x(q-1)+(q^2-1)+l(q+1)}{2}$ implies $q^{\frac{y(q-1)+l(q+1)}{2}-(j_{\max}(q-1)+l-q)}\cdot q^{(x+1)(q-1)+l}<q^{y(q-1)+l(q+1)-(j_{\max}(q-1)+l-q)}$. Therefore, we can conclude

$$|a_{T_{\mathfrak{p}}(g_1^y h^l+cg_1^x\Delta h^l)}((q-1)+l)|<q^{y(q-1)+l(q+1)-(j_{\max}(q-1)+l-q)},$$

which contradicts (4.16). Hence, the $T_{\mathfrak{p}}$-operator has no eigenform on $S_{k,l}(\mathrm{GL}_2(A))$ with eigenvalue $\pm P^{k/2}$, and the result now follows from Proposition 4.5.

We now consider the case when $(y,l)=(2q+1,1)$. In this case, $S_{k,l}(\mathrm{GL}_2(A))=\langle g_1^{2q+1}h,g_1^q\Delta h\rangle$. By the u-series expansion, we get $a_{T_{\mathfrak{p}}(g_1^{2q+1}h)}(1)=-P,a_{T_{\mathfrak{p}}(g_1^q\Delta h)}(1)=0$ and $a_{T_{\mathfrak{p}}(g_1^q\Delta h)}(q)=P^q$. This implies that, for any $(c_1,c_2)\in C^2\backslash\{(0,0)\}$, $T_{\mathfrak{p}}(c_1g_1^{2q+1}h+c_2g_1^q\Delta h)\ne P^{q^2}(c_1g_1^{2q+1}h+c_2g_1^q\Delta h)$. This can be checked by comparing the 1-st coefficient if $c_1\ne0$, the q-th coefficient if $c_1=0$. Now, we are done by Proposition 4.5.

Finally, consider the case $l=0$: Here, the proof is exactly similar to $l\ne0$, except that we need to consider $(q-1)$-th, $2(q-1)$-th coefficients and use the inequality $(x+2)(q-1)<\frac{x(q-1)+2(q^2-1)}{2}$. □

4.2. **Evidences to Question 4.3 for prime ideals $\mathfrak{p}$:** In this section, we give evidences in the support of Question 4.3 for prime ideals $\mathfrak{p}$. In this direction, we need a proposition, which is a generalization of a result of Gekeler (cf. [Gek88, Corollary 7.6]), where he proved that $T_{\mathfrak{p}}h=Ph$ for any prime ideal $\mathfrak{p}=(P)$. We now show that this result continues to hold for $f\in M_{k,1}(\Gamma_0(\mathfrak{m}))$ with $a_f(1)\ne0$.

Proposition 4.8. *Suppose the u-series expansion of $f\in M_{k,1}(\Gamma_0(\mathfrak{m}))$ at ∞ is given by $\sum_{j=0}^{\infty}a_f(j(q-1)+1)u^{j(q-1)+1}$ with $a_f(1)\ne0$. If $T_{\mathfrak{p}}f=\lambda f$ for some $\lambda\in C$, then $\lambda=P$. In particular, $T_{\mathfrak{p}}f=P^{\frac{k}{2}}f$ can happen only when $k=2$.*

Proof. Let $G_{i,P}(X)$ denote the i-th Goss polynomial corresponding to the lattice $\Lambda_P=\ker(\rho_P)=\{x\in C\mid\rho_P(x)=0\}$, where ρ_P is the Carlitz module with value at P. By [Arm11, Proposition 5.2] (the normalization here is different from there), we have

$$T_{\mathfrak{p}}f=P^k\sum_{j\ge0}a_f(j(q-1)+1)(u_{\mathfrak{p}})^{j(q-1)+1}+\sum_{j\ge0}a_f(j(q-1)+1)G_{j(q-1)+1,P}(Pu), \tag{4.17}$$

where $u_{\mathfrak{p}}(z)=u(Pz)=u^{q^d}+\cdots$. To determine λ, we wish to compute the coefficient of u in the u-series expansion of $T_{\mathfrak{p}}f$. In (4.17), the term involving $u_{\mathfrak{p}}$ does not contribute to the coefficient of u. By [Gek88, Proposition 3.4(ii)], we know that

$$G_{i,P}(X)=X(G_{i-1,P}(X)+\alpha_1G_{i-q,P}(X)+\cdots+\alpha_jG_{i-q^j,P}(X)+\cdots).$$

In $G_{i,P}(Pu)$, the coefficient of u in $G_{j(q-1)+1,P}(Pu)$ is 0 for $j>0$. Since $G_{1,P}(X)=X$ (cf. [Gek88, Proposition 3.4(v)]), we can conclude that $T_{\mathfrak{p}}f=Pa_f(1)u+$ higher terms. By comparing the coefficient of u on both sides, we get $\lambda=P$. □

Remark 4.9. In Proposition 4.8, Goss polynomials, which occur as the coefficients of $T_\mathfrak{p} f$, are very difficult to handle if $l \neq 1$ (cf. (4.17) and [Arm11, Proposition 5.2]). So, we have restricted ourselves to $l = 1$ in the last proposition.

We now give some instances where Question 4.3 for prime ideals $\mathfrak{p}$ has an affirmative answer.

Proposition 4.10. *For any prime ideal $\mathfrak{p}$, Question 4.3 is true for level $\mathfrak{p}$ in the following cases:*

(1) (a) $1 \leq l \leq q-2$ and $k = 2l + \alpha(q-1)$ where $\alpha \in \{0, \dots, l\}$,
(b) $l = 0$ and $k = \beta(q-1)$ where $\beta \in \{1, \dots, q+1\}$,
(c) $l = 1$ and $k = \alpha(q-1) + (q+1)$ where $\alpha \in \{0, \dots, q\}$.

(2) $k \leq 3q$.

Proof. Note that in all of these cases $\dim S_{k,l}(\mathrm{GL}_2(A)) \leq 1$. Hence the $T_\mathfrak{p}$-operator is diagonalizable on $S_{k,l}(\mathrm{GL}_2(A))$. As in our earlier discussion, Question 4.3 has an affirmative answer for $\mathfrak{p}$ if we show that $\ker(T_\mathfrak{p}) = 0$ and the $T_\mathfrak{p}$-operator has no eigenform on $S_{k,l}(\mathrm{GL}_2(A))$ with eigenvalues $\pm P^{\frac{k}{2}}$. We prove these statements in all cases.

(1) (a) Since $l > 0$, $M_{k,l}(\mathrm{GL}_2(A)) = S_{k,l}(\mathrm{GL}_2(A))$.
- If $\alpha \in \{0, \dots, l-1\}$, then $\dim S_{2l+\alpha(q-1),l}(\mathrm{GL}_2(A)) = 0$ and the result follows trivially.
- If $\alpha = l$, then $\dim S_{2l+\alpha(q-1),l}(\mathrm{GL}_2(A)) = 1$ and $S_{2l+\alpha(q-1),l}(\mathrm{GL}_2(A)) = \langle h^l \rangle$. By [JP14, (9)] (or by [Pet13, Theorem 3.17]), the $T_\mathfrak{p}$-operator acts on h^i by P^i for $1 \leq i \leq q-2$. Since $P^l \neq \pm P^{\frac{l(q+1)}{2}}$ for $1 \leq l \leq q-2$ the result follows.

(b) When $l = 0$, we prove the required claim in two steps.
- For $\beta \in \{1, \dots, q\}$, $M_{\beta(q-1),0}(\mathrm{GL}_2(A)) = \langle g_1^\beta \rangle$. Therefore, $S_{\beta(q-1),0}(\mathrm{GL}_2(A)) = \{0\}$ and the result follows.
- If $\beta = q+1$, $S_{q^2-1,0}(\mathrm{GL}_2(A)) = \langle \Delta \rangle$. By [Gek88, Corollary 7.5], we have $T_\mathfrak{p}(\Delta) = P^{q-1}\Delta$. Since $P^{q-1} \neq \pm P^{\frac{q^2-1}{2}}$, the result follows.

(c) If $\alpha \in \{0, \dots, q\}$, $S_{k,1}(\mathrm{GL}_2(A)) = \langle g_1^\alpha h \rangle$. Since $a_{g_1^\alpha h}(1) \neq 0$, by Proposition 4.8, we have $T_\mathfrak{p}(g_1^\alpha h) = P g_1^\alpha h$, the result follows.

(2) Let $0 \leq l \leq q-2$. If $k \not\equiv 2l \pmod{q-1}$, then $M_{k,l}(\mathrm{GL}_2(A)) = \{0\}$ and Question 4.3 is trivially true. So, we only consider the cases $k \equiv 2l \pmod{q-1}$ i.e $k = 2l + x(q-1)$ for some $x \in \mathbb{N} \cup \{0\}$. The condition $k \leq 3q$ implies $x \leq 4$.
- If $x < l$, then $\dim M_{k,l}(\mathrm{GL}_2(A)) = 0$ and the result follows.
- If $x = l$, then $k = l(q+1)$. If $l \neq 0$, then $S_{l(q+1),l}(\mathrm{GL}_2(A)) = \langle h^l \rangle$. So, we are back to case 1(a). If $l = 0$, then $S_{0,0} = \{0\}$ and the result follows.

Therefore, the remaining cases of interest are $l < x \leq 4$. If $l \geq 2$, the inequality $k \leq 3q$ forces that $x \leq 2$ and we are back to the case $x \leq l$. So, it is enough to consider for $l \in \{0, 1\}$ with $l < x \leq 4$.
- For $l = 0$: If $(q, x) \neq (3, 4)$, then $M_{x(q-1),0}(\mathrm{GL}_2(A)) = \langle g_1^x \rangle$ and $S_{x(q-1),0}(\mathrm{GL}_2(A)) = \{0\}$ and the result follows. If $(q, x) = (3, 4)$, then $k = (q+1)(q-1)$, we are back to case $1(b)$.
- For $l = 1$: we have $k = (x-1)(q-1) + (q+1)$ where $1 < x \leq 3$. Since $q \geq 3$, we are back to case $1(c)$.

This completes the proof of the proposition. □

We remark that our Proposition 4.10 is similar to Theorem 5.8, Corollary 5.11 and Theorem 5.14 of [BV19a] for $\mathfrak{p} = (T)$-case. In a contrast to Proposition 4.10, in the next proposition, we consider the situation with $\mathfrak{m} \neq A$ and $\dim S_{k,l}(\Gamma_0(\mathfrak{m})) = 2$ satisfying Question 4.3 for level $\mathfrak{pm}$.

Proposition 4.11. *For $\deg m = 1$ and $\mathfrak{p} \nmid \mathfrak{m}$, Question 4.3 is true for level $\mathfrak{pm}$ when*

(i) $l > \frac{q-1}{2}$ and $k = 2l - (q-1)$, or
(ii) $l = 1$ and $k = q+1$.

Proof. We may assume that $\mathfrak{m} = (T)$, since a similar calculation works for any ideal $\mathfrak{m}$ with $\deg m = 1$. We now follow the strategy as in the proof of the Proposition 4.10.

(i) In this case, $S_{k,l}(\Gamma_0(T)) = \{0\}$ (cf. [DK, Proposition 4.1]) and the result follows trivially.
(ii) First, we show that the operator $T_\mathfrak{p} - P$ is zero on $S_{q+1,1}(\Gamma_0(T))$. Recall that, $\Delta_T(z) := \frac{g_1(Tz) - g_1(z)}{T^q - T}$, $\Delta_W(z) := \frac{T^q g_1(Tz) - T g_1(z)}{T^q - T} \in M_{q-1,0}(\Gamma_0(T))$.

By [DK, Proposition 4.3(3)], $\dim_C\ S_{q+1,1}(\Gamma_0(T)) = 2$ and a basis is given by $\{\Delta_T E_T, \Delta_W E_T\}$. By [DK, Proposition 4.3(8)]), $h = -\Delta_W E_T$. Since $T_\mathfrak{p} h = Ph$, we obtain $T_\mathfrak{p}(\Delta_W E_T) = P\Delta_W E_T$. Note that $\Delta_T = -T^{-1}\Delta_W|W_T^{(T)}$ and $T_\mathfrak{p} W_T^{(T)} = W_T^{(T)} T_\mathfrak{p}$ (cf. Theorem 5.3), using $E_T|_{2,1}W_T^{(T)} = -E_T$ (cf. [DK21, Proposition 3.3]), we get

$$\begin{aligned} T_\mathfrak{p}(\Delta_T E_T) &= T_\mathfrak{p}((T^{-1}\Delta_W E_T)|W_T^{(T)}) = (T_\mathfrak{p}(T^{-1}\Delta_W E_T))|W_T^{(T)} \\ &= T^{-1}(P\Delta_W E_T)|W_T^{(T)} = P\Delta_T E_T. \end{aligned}$$

Thus, $T_\mathfrak{p} \equiv P$ on $S_{q+1,1}(\Gamma_0(T))$. So, $T_\mathfrak{p}$-operator is injective, diagonalizable on $S_{q+1,1}(\Gamma_0(T))$, which proves Question 4.3(1). Question 4.3(2) follows from Proposition 4.5. Finally, Question 4.3(3) follows from the diagonalizability of $T_\mathfrak{p}$-operator on $S_{q+1,1}(\Gamma_0(T))$.

□

4.3. **Counterexample to Question 4.3(2).** In the section, we show that the direct sum decomposition (4.1) does not hold if $\mathfrak{m} \neq A$ and $(k,l) = (2,1)$. We first prove a result which is of independent interest.

Lemma 4.12. *Let $\mathfrak{p}_1, \mathfrak{p}_2$ be two distinct prime ideals of A generated by monic irreducible polynomials P_1, P_2, respectively. Then, $T_{\mathfrak{p}_1} E_{P_2} = P_1 E_{P_2}$.*

Proof. By [Gek88, (8.2)], the function $E(z) = \sum_{a\in A_+} au(az)$, where A_+ denotes the set of all monic polynomials in A. Hence, $E_{P_2}(z) = \sum_{a\in A_+} au(az) - P_2\sum_{a\in A_+} au(P_2az) = \sum_{a\in A_+,\ P_2\nmid a} au(az)$. We now use an argument in the proof of [Pet13, Theorem 2.3] to get

$$\begin{aligned} T_{\mathfrak{p}_1}E_{P_2} &= \sum_{\substack{Q\in A\\ \deg Q<\deg P_1}} E_{P_2}\left(\frac{z+Q}{P_1}\right) + P_1^2 E_{P_2}(P_1 z) \\ &= \sum_{\substack{Q\in A\\ \deg Q<\deg P_1}} \sum_{\substack{a\in A_+\\ P_2\nmid a}} au\left(a\frac{z+Q}{P_1}\right) + P_1^2 \sum_{\substack{a\in A_+\\ P_2\nmid a}} au(P_1az) \\ &= \frac{1}{\tilde{\pi}} \sum_{\substack{Q\in A\\ \deg Q<\deg P_1}} \sum_{\substack{a\in A_+\\ P_2\nmid a}} \sum_{b\in A} \frac{aP_1}{az+aQ+P_1b} + P_1 \sum_{\substack{a\in A_+\\ P_2\nmid a}} P_1au(P_1az) \\ &= \frac{1}{\tilde{\pi}} \sum_{\substack{a\in A_+\\ P_2\nmid a}} aP_1 \sum_{b\in A} \sum_{\substack{Q\in A\\ \deg Q<\deg P_1}} \frac{1}{az+aQ+P_1b} + P_1 \sum_{\substack{a\in A_+\\ P_2\nmid a}} P_1au(P_1az) \\ &= P_1 \sum_{\substack{a\in A_+\\ P_1P_2\nmid a}} au(az) + P_1 \sum_{\substack{a\in A_+\\ P_2\nmid a}} P_1au(P_1az) = P_1 \sum_{\substack{a\in A_+\\ P_2\nmid a}} au(az) = P_1E_{P_2}. \end{aligned}$$

This completes the proof of the Lemma. □

We now show that, if $\mathfrak{m} \neq A$ and $(k,l) = (2,1)$, then there are non-zero Drinfeld cusp forms which are both $\mathfrak{p}$-oldforms and $\mathfrak{p}$-newforms.

Proposition 4.13. *Suppose $\mathfrak{m} \neq A$. For any prime ideal $\mathfrak{p} \nmid \mathfrak{m}$, we have*

$$S_{2,1}^{\mathfrak{p}-\mathrm{old}}(\Gamma_0(\mathfrak{pm})) \cap S_{2,1}^{\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{pm})) \neq \{0\}.$$

Proof. Let $\mathfrak{p}_2$ be a prime divisor of $\mathfrak{m}$ generated by a monic irreducible polynomial P_2. Clearly, $0 \neq E_{P_2} - \delta_P E_{P_2} \in S_{2,1}(\Gamma_0(\mathfrak{pm}))$. We show that $E_{P_2} - \delta_P E_{P_2} \in S_{2,1}^{\mathfrak{p}-\mathrm{old}}(\Gamma_0(\mathfrak{pm})) \cap S_{2,1}^{\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{pm}))$. By definition, $E_{P_2} - \delta_P E_{P_2} \in S_{2,1}^{\mathfrak{p}-\mathrm{old}}(\Gamma_0(\mathfrak{pm}))$. Combining (4.3), (4.2) and Lemma 4.12, we get

$$\mathrm{Tr}_\mathfrak{m}^{\mathfrak{pm}}(E_{P_2} - \delta_P E_{P_2}) = E_{P_2} - P^{-1}T_\mathfrak{p}(E_{P_2}) = E_{P_2} - E_{P_2} = 0. \tag{4.18}$$

By (4.2), we deduce that

$$\mathrm{Tr}_\mathfrak{m}^{\mathfrak{pm}}((E_{P_2} - \delta_P E_{P_2})|W_\mathfrak{p}^{(\mathfrak{pm})}) = \mathrm{Tr}_\mathfrak{m}^{\mathfrak{pm}}(E_{P_2}|W_\mathfrak{p}^{(\mathfrak{pm})} - (\delta_P E_{P_2})|W_\mathfrak{p}^{(\mathfrak{pm})}) = \mathrm{Tr}_\mathfrak{m}^{\mathfrak{pm}}(\delta_P E_{P_2} - E_{P_2}) = 0.$$

This proves that $E_{P_2} - \delta_P E_{P_2} \in S_{2,1}^{\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{pm}))$. The result follows. □

Remark 4.14. For $f \in S_{k,l}(\Gamma_0(\mathfrak{n}))$, $T_{\mathfrak{p}}(f^{q^n}) = (T_{\mathfrak{p}}(f))^{q^n}$ for any $n \in \mathbb{N}$. An argument similar to Proposition 4.13 gives us

$$0 \neq E_{P_2}^{q^n} - P^{q^n-1}\delta_P E_{P_2}^{q^n} \in S_{2q^n,1}^{\mathfrak{p}-\mathrm{old}}(\Gamma_0(\mathfrak{pm})) \cap S_{2q^n,1}^{\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{pm})). \tag{4.19}$$

Since E behaves like a classical weight 2 Eisenstein series, we believe that the phenomenon in (4.19) may not happen for $l \neq 1$.

Proposition 4.13 and Remark 4.14 imply that either one needs to reformulate the definition of $\mathfrak{p}$-newforms for level $\mathfrak{pm}$ or exclude the cases above in formulating Question 4.3 for level $\mathfrak{pm}$.

5. Oldforms and Newforms for square-free level $\mathfrak{n}$

In this section, we propose a definition of oldforms and newforms for Drinfeld modular forms of square-free level. We show that these spaces are invariant under the action of the Hecke operators. Throughout this section, we assume that $\mathfrak{n}$ is a square-free ideal of A generated by a (square-free) monic polynomial $n \in A$. Let $\mathfrak{p}, \mathfrak{p}_1$ be two prime ideals of A generated by monic irreducible polynomials $P, P_1 \in A$, respectively.

Definition 5.1 (Oldforms)**.** The space of oldforms of weight k, type l, and square-free level $\mathfrak{n}$ is defined as

$$S_{k,l}^{\mathrm{old}}(\Gamma_0(\mathfrak{n})) := \sum_{\mathfrak{p}|\mathfrak{n}} (\delta_1, \delta_P)((S_{k,l}(\Gamma_0(\mathfrak{n}/\mathfrak{p})))^2).$$

The lack of Petersson inner product for Drinfeld modular forms makes it difficult to define newforms. For classical modular forms, it is well-known that newforms can be characterized in terms of kernels of the Trace and twisted Trace operators (cf. [Ser73], [Li75] for more details). In this section, for Drinfeld modular forms, we adopt this approach to define newforms and investigate their properties.

Definition 5.2 (Newforms)**.** The space of newforms of weight k, type l, and square-free level $\mathfrak{n}$ is defined as

$$S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n})) := \bigcap_{\mathfrak{p}|\mathfrak{n}} (\mathrm{Ker}(\mathrm{Tr}^{\mathfrak{n}}_{\frac{\mathfrak{n}}{\mathfrak{p}}}) \cap \mathrm{Ker}(\mathrm{Tr}'^{\mathfrak{n}}_{\frac{\mathfrak{n}}{\mathfrak{p}}})), \text{ where } \mathrm{Tr}'^{\mathfrak{n}}_{\frac{\mathfrak{n}}{\mathfrak{p}}} f = \mathrm{Tr}^{\mathfrak{n}}_{\frac{\mathfrak{n}}{\mathfrak{p}}}(f|_{k,l} W_{\mathfrak{p}}^{(\mathfrak{n})}).$$

Next, we study the action of Hecke operators on $S_{k,l}^{\mathrm{old}}(\Gamma_0(\mathfrak{n}))$, $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$. This depends on the commutativity of the (partial) Atkin-Lehner operators with the $T_{\mathfrak{p}}$ and $U_{\mathfrak{p}}$-operators. In [Val22, Theorem 1.1], the author studied the commutativity of the (partial) Atkin-Lehner operators and the $T_{\mathfrak{p}}$-operator and proved the following result.

Theorem 5.3. *Let $\mathfrak{n}, \mathfrak{p} \subseteq A$ be ideals such that $\mathfrak{p} \nmid \mathfrak{n}$ and $\mathfrak{p}$ is prime. For any ideal $\mathfrak{d}$ of A such that $\mathfrak{d}||\mathfrak{n}$, the actions of $T_{\mathfrak{p}} W_{\mathfrak{d}}^{(\mathfrak{n})}$ and $W_{\mathfrak{d}}^{(\mathfrak{n})} T_{\mathfrak{p}}$ on $S_{k,l}(\Gamma_0(\mathfrak{n}))$ are equal.*

We now study the commutativity of certain (partial) Atkin-Lehner operators and the $U_{\mathfrak{p}}$-operator. The following result can be thought of as a generalization of Theorem 5.3 to the $U_{\mathfrak{p}}$-operator. Note that, Theorem 5.4 holds for any integral ideal $\mathfrak{n}$.

Theorem 5.4. *Assume that $\mathfrak{p}^\alpha||\mathfrak{n}$ for some $\alpha \in \mathbb{N}$. For all prime divisors $\mathfrak{p}_1$ of $\mathfrak{n}$ with $\mathfrak{p}_1 \neq \mathfrak{p}$, the actions of $U_{\mathfrak{p}_1} W_{\mathfrak{p}^\alpha}^{(\mathfrak{n})}$ and $W_{\mathfrak{p}^\alpha}^{(\mathfrak{n})} U_{\mathfrak{p}_1}$ on $S_{k,l}(\Gamma_0(\mathfrak{n}))$ are equal.*

Proof. By definition we have

$$P_1^{l-k} U_{\mathfrak{p}_1} W_{\mathfrak{p}^\alpha}^{(\mathfrak{n})} = \sum_{\substack{Q \in A \\ \deg Q < \deg P_1}} \begin{pmatrix} P^\alpha & b \\ n & P^\alpha d \end{pmatrix} \begin{pmatrix} 1 & Q \\ 0 & P_1 \end{pmatrix} = \sum_{\substack{Q \in A \\ \deg Q < \deg P_1}} \begin{pmatrix} P^\alpha & P^\alpha Q + bP_1 \\ n & nQ + P^\alpha P_1 d \end{pmatrix},$$

$$P_1^{l-k} W_{\mathfrak{p}^\alpha}^{(\mathfrak{n})} U_{\mathfrak{p}_1} = \sum_{\substack{Q \in A \\ \deg Q < \deg P_1}} \begin{pmatrix} 1 & Q \\ 0 & P_1 \end{pmatrix} \begin{pmatrix} P^\alpha & b \\ n & P^\alpha d \end{pmatrix} = \sum_{\substack{Q \in A \\ \deg Q < \deg P_1}} \begin{pmatrix} P^\alpha + Qn & b + P^\alpha Q d \\ nP_1 & P^\alpha P_1 d \end{pmatrix}.$$

To prove the proposition, it suffices to show that for any $Q \in A$ with $\deg Q < \deg P_1$, there exists a unique $Q' \in A$ with $\deg Q' < \deg P_1$ such that

$$\begin{pmatrix} P^\alpha + Qn & b + P^\alpha Q d \\ nP_1 & P^\alpha P_1 d \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} P^\alpha & P^\alpha Q' + bP_1 \\ n & nQ' + P^\alpha P_1 d \end{pmatrix}, \tag{5.1}$$

for some $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{n})$. For any $Q, Q' \in A$, (5.1) implies $x, w \in A$, $z \in \mathfrak{n}$ and

$$-P_1 y = P^\alpha Q' - P^\alpha Q d - b + (nQQ' + bP_1 + \frac{n}{P^\alpha} bP_1 Q). \tag{5.2}$$

Thus we are reduced to show that for any $Q \in A$ with $\deg Q < \deg P_1$, there exists a unique $Q' \in A$ with $\deg Q' < \deg P_1$ such that $y \in A$.

Since $P_1 | n$, we have $P_1 | (nQQ' + bP_1 + \frac{n}{P^\alpha} bP_1 Q)$. Now it is enough to show that there exists a unique $Q' \in A$ with $\deg Q' < \deg P_1$ such that $P_1 \mid P^\alpha(Q' - Qd) - b$.

Recall that, $P^\alpha d - b\frac{n}{P^\alpha} = 1$. Since P_1 divides $\frac{n}{P^\alpha}$, we get $QP^\alpha d \equiv Q \pmod{P_1}$ for any $Q \in A$. So, it is enough to show that there exists a unique $Q' \in A$ such that $P_1 \mid P^\alpha Q' - (Q + b)$. Since $(P^\alpha, P_1) = 1$, the congruence $P^\alpha f(X) \equiv (Q + b) \pmod{P_1}$ has a unique solution in A with $\deg(f(X)) < \deg P_1$. We are done. ☐

We are now ready to state the main theorem of this section.

Theorem 5.5. *The spaces $S_{k,l}^{\mathrm{old}}(\Gamma_0(\mathfrak{n}))$, $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ are invariant under the action of the Hecke operators $T_\mathfrak{p}$ for $\mathfrak{p} \nmid \mathfrak{n}$ and $U_\mathfrak{p}$ for $\mathfrak{p} \mid \mathfrak{n}$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \mid \mathfrak{n}$. We first show that the space $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ is stable under the $U_\mathfrak{p}$-operator. Let $\mathfrak{p}_1 \neq \mathfrak{p}$ be a prime divisor of $\mathfrak{n}$ and $f \in S_{k,l}^{\mathfrak{p}_1-\mathrm{new}}(\Gamma_0(\mathfrak{n}))$. Theorem 5.4 (resp., Proposition 3.6) implies that the $U_\mathfrak{p}$-operator commutes with the $W_{\mathfrak{p}_1}^{(\mathfrak{n})}$-operator (resp., the $U_{\mathfrak{p}_1}$-operator). Since $f \in S_{k,l}^{\mathfrak{p}_1-\mathrm{new}}(\Gamma_0(\mathfrak{n}))$, from Proposition 3.8, we obtain

$$\mathrm{Tr}^{\mathfrak{n}}_{\frac{\mathfrak{n}}{\mathfrak{p}_1}}(U_\mathfrak{p}(f)) = U_\mathfrak{p}(f) + P_1^{-l} U_{\mathfrak{p}_1}(U_\mathfrak{p}(f)|W_\mathfrak{p}^{(\mathfrak{n})}) = U_\mathfrak{p}(\mathrm{Tr}^{\mathfrak{n}}_{\frac{\mathfrak{n}}{\mathfrak{p}_1}}(f)) = 0.$$

A similar argument shows that $\mathrm{Tr}'^{\mathfrak{n}}_{\frac{\mathfrak{n}}{\mathfrak{p}_1}}(U_\mathfrak{p}(f)) = 0$. Thus $S_{k,l}^{\mathfrak{p}_1-\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ is stable under the $U_\mathfrak{p}$-operator. Since the space $S_{k,l}^{\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ is stable under the action of the $U_\mathfrak{p}$-operator (cf. [BV20, Proposition 2.15]), the space $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ is stable under the action of the $U_\mathfrak{p}$-operator.

Next, we show that the space $S_{k,l}^{\mathrm{old}}(\Gamma_0(\mathfrak{n}))$ is stable under the action of the $U_\mathfrak{p}$-operator. Let $\mathfrak{p}_1 \neq \mathfrak{p}$ be a prime divisor of $\mathfrak{n}$. Let $\psi, \varphi \in S_{k,l}(\Gamma_0(\frac{\mathfrak{n}}{\mathfrak{p}_1}))$. Since $\mathfrak{p} | \frac{\mathfrak{n}}{\mathfrak{p}_1}$, we have $U_\mathfrak{p}(\psi), U_\mathfrak{p}(\varphi) \in S_{k,l}(\Gamma_0(\frac{\mathfrak{n}}{\mathfrak{p}_1}))$. Moreover, (4.2) and Theorem 5.4 yield

$$U_\mathfrak{p}(\delta_{P_1}\varphi) = U_\mathfrak{p}(\varphi|W_{P_1}^{(\mathfrak{n})}) = (U_\mathfrak{p}(\varphi))|W_{P_1}^{(\mathfrak{n})} = \delta_{P_1}(U_\mathfrak{p}(\varphi)).$$

Hence for all $\mathfrak{p}_1 \mid \mathfrak{n}$ with $\mathfrak{p}_1 \neq \mathfrak{p}$ we have $U_\mathfrak{p}(\psi + \delta_{P_1}\varphi) = U_\mathfrak{p}(\psi) + \delta_{P_1} U_\mathfrak{p}(\varphi)$ with $U_\mathfrak{p}(\psi), U_\mathfrak{p}(\varphi) \in S_{k,l}(\Gamma_0(\frac{\mathfrak{n}}{\mathfrak{p}_1}))$. Since the space $S_{k,l}^{\mathfrak{p}-\mathrm{old}}(\Gamma_0(\mathfrak{n}))$ is stable under the action of the $U_\mathfrak{p}$-operator (cf. [BV20, Proposition 2.15]), the space $S_{k,l}^{\mathrm{old}}(\Gamma_0(\mathfrak{n}))$ is stable under the action of the $U_\mathfrak{p}$-operator.

An argument similar to the above would also imply that the spaces $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ and $S_{k,l}^{\mathrm{old}}(\Gamma_0(\mathfrak{n}))$ are stable under the $T_\mathfrak{p}$-operator for $\mathfrak{p} \nmid \mathfrak{n}$. ☐

Corollary 5.6. *The set of $U_\mathfrak{p}$-operators (for $\mathfrak{p} \mid \mathfrak{n}$) are simultaneously diagonalizable on $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$.*

Proof of Corollary 5.6. Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \mid \mathfrak{n}$. By [BV20, Remark 2.17], the $U_\mathfrak{p}$-operator is diagonalizable on $S_{k,l}^{\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{n}))$. By Theorem 5.5, the space $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ is an $U_\mathfrak{p}$-invariant subspace of $S_{k,l}^{\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{n}))$, hence the $U_\mathfrak{p}$-operator is also diagonalizable on $S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$. Now, the corollary follows from Proposition 3.6 and the fact that a commuting set of diagonalizable operators on a finite dimensional vector space are simultaneously diagonalizable. ☐

We conclude this article with a remark that $S_{k,l}^{\mathrm{old}}(\Gamma_0(\mathfrak{n})) \cap S_{k,l}^{\mathrm{new}}(\Gamma_0(\mathfrak{n})) = \{0\}$ may happen only for $l \neq 1$, because of the following proposition, which is in the spirit of Proposition 4.13. As a result, one may have to reformulate the definition of oldforms and newforms of level $\mathfrak{pm}$ for $l = 1$.

Proposition 5.7. *For any two distinct prime ideals $\mathfrak{p}, \mathfrak{q}$ generated by monic irreducible polynomials P, Q, respectively, the intersection $S_{2,1}^{\mathrm{old}}(\Gamma_0(\mathfrak{pq})) \cap S_{2,1}^{\mathrm{new}}(\Gamma_0(\mathfrak{pq})) \neq \{0\}$. Furthermore, for any $x \in \mathbb{N}$, $S_{2q^x,1}^{\mathrm{old}}(\Gamma_0(\mathfrak{pq})) \cap S_{2q^x,1}^{\mathrm{new}}(\Gamma_0(\mathfrak{pq})) \neq \{0\}$.*

Proof. We now show that $E_Q - \delta_P E_Q \in S_{2,1}^{\mathrm{old}}(\Gamma_0(\mathfrak{pq})) \cap S_{2,1}^{\mathrm{new}}(\Gamma_0(\mathfrak{pq}))$. By definition, $0 \neq E_Q - \delta_P E_Q \in S_{2,1}^{\mathrm{old}}(\Gamma_0(\mathfrak{pq}))$. From (4.18) and (4.3), we have $\mathrm{Tr}^{\mathfrak{pq}}_{\mathfrak{q}}(E_Q - \delta_P E_Q) = 0 = \mathrm{Tr}^{\mathfrak{pq}}_{\mathfrak{q}}((E_Q -$

$\delta_P E_Q)|W_{\mathfrak{p}}^{(\mathfrak{pq})})$. Since $W_{\mathfrak{p}}^{(\mathfrak{pq})}W_{\mathfrak{q}}^{(\mathfrak{pq})} = W_{\mathfrak{q}}^{(\mathfrak{pq})}W_{\mathfrak{p}}^{(\mathfrak{pq})}$, $W_{\mathfrak{p}}^{(\mathfrak{pq})}U_{\mathfrak{q}} = U_{\mathfrak{q}}W_{\mathfrak{p}}^{(\mathfrak{pq})}$, by Proposition 3.8 and (4.2), we get

$$\begin{aligned}\mathrm{Tr}_{\mathfrak{p}}^{\mathfrak{pq}}(E_Q - \delta_P E_Q) &= \mathrm{Tr}_{\mathfrak{p}}^{\mathfrak{pq}}(E_Q) - \mathrm{Tr}_{\mathfrak{p}}^{\mathfrak{pq}}(E_Q)|W_{\mathfrak{p}}^{(\mathfrak{pq})} \\ &= \mathrm{Tr}_1^{\mathfrak{q}}(E_Q) - \mathrm{Tr}_1^{\mathfrak{q}}(E_Q)|W_{\mathfrak{p}}^{(\mathfrak{pq})} \quad \text{(cf. Corollary 3.9)} \\ &= 0 \qquad \text{(since } M_{2,1}(\mathrm{GL}_2(A)) = 0).\end{aligned}$$

Since $E_Q|W_{\mathfrak{q}}^{(\mathfrak{pq})} = E_Q|W_{\mathfrak{q}}^{(\mathfrak{q})} = -E_Q$ (cf. [DK21, Proposition 3.3])), we have

$$\begin{aligned}\mathrm{Tr}_{\mathfrak{p}}^{\mathfrak{pq}}((E_Q - \delta_P E_Q)|W_{\mathfrak{q}}^{(\mathfrak{pq})}) &= \mathrm{Tr}_{\mathfrak{p}}^{\mathfrak{pq}}((E_Q|W_{\mathfrak{q}}^{(\mathfrak{pq})}) - E_Q|W_{\mathfrak{p}}^{(\mathfrak{pq})}W_{\mathfrak{q}}^{(\mathfrak{pq})}) \\ &= \mathrm{Tr}_{\mathfrak{p}}^{\mathfrak{pq}}((E_Q|W_{\mathfrak{q}}^{(\mathfrak{pq})}) - (E_Q|W_{\mathfrak{q}}^{(\mathfrak{pq})})|W_{\mathfrak{p}}^{(\mathfrak{pq})}) \\ &= \mathrm{Tr}_{\mathfrak{p}}^{\mathfrak{pq}}(-E_Q + (E_Q|W_{\mathfrak{p}}^{(\mathfrak{pq})})) = \mathrm{Tr}_{\mathfrak{p}}^{\mathfrak{pq}}(-E_Q + \delta_P E_Q) = 0.\end{aligned}$$

Hence, $E_Q - \delta_P E_Q \in S_{2,1}^{\mathrm{new}}(\Gamma_0(\mathfrak{pq}))$. A similar argument shows that $0 \neq E_Q^{q^x} - P^{q^x-1}\delta_P E_Q^{q^x} \in S_{2q^x,1}^{\mathrm{old}}(\Gamma_0(\mathfrak{pq})) \cap S_{2q^x,1}^{\mathrm{new}}(\Gamma_0(\mathfrak{p}_1\mathfrak{q}))$ for $x \in \mathbb{N}$. □

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