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Nonlocal nonlinear bending and free vibration analysis of a rotating laminated nano cantilever beam

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Abstract

In this paper, we present the non-local nonlinear finite element formulations for the case of nonuniform rotating laminated nano cantilever beams using the Timoshenko beam theory. The surface stress effects are also taken into consideration. Non-local stress resultants are obtained by employing Eringen's nonlocal differential model. Geometric nonlinearity is taken into account by using the Green Lagrange strain tensor. Numerical solutions of nonlinear bending and free vibration are presented. Parametric studies have been carried out to understand the effect of non-local parameter and surface stresses on bending and vibration behavior of cantilever beams. Also, the effects of angular velocity and hub radius on the vibration behavior of the cantilever beam are studied.

Keywords: Timoshenko beam, nonlocal parameter, surface stress, geometric non-linearity, finite element analysis, numerical results.

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1 Introduction

The properties of any material at the macroscopic scale are influenced by the inhomogeneities present at the microscopic scale and nanoscale. This points to the need for incorporating micro motions in continuum mechanical formulations [1] and [2]. There has been considerable focus towards the development of generalized continuum theories [3], which account for the inherent microstructure in such natural and engineering materials (see [4] and [5]).The notion of generalized continua unifies several extended continuum theories that account for such a size dependence due to the underlying microstructure of the material. A systematic overview and detailed discussion of generalized continuum theories has been given by Bazant and Jirasek [6]. These theories can be categorized as gradient continuum theories for instance see works of Mindilin et al. ([7],[8], and [9]), Toupin[10], Steinmann et al. ([1], [11],[12], and [13]), and works of Casterzene et al.[14], Fleck et al.([15] and [16]), Askes et al.([17], [18], and[19]), micro continuum theories (see works by Eringen[20],[21],[3]), Steinmann et al. [22], [23], and nonlocal continuum theories (see works by Eringen[24], Jirasek [25], Reddy [26], and others [27], [28] and [29], [30]). In some of the earlier works, the higher-order gradient theory for finite deformation has been elaborated (for instance see[31], [32],[33], and [14]) within classical continuum mechanics in the context of homogenization approaches. A comparison of various higher order gradient theories can be found in [15]. A more detailed formulation of gradient approach in spatial and material setting has been presented in [22] and an overview of nonlocal theories of continuum mechanics can be found in [34].

The nonlocal formulations can be of integral-type with weighted spatial averaging or by implicit gradient models which are categorized as strongly nonlocal, while weakly nonlocal theories include, for instance, explicit gradient models [6]. In this work, we consider a strongly nonlocal problem. The Timoshenko beam can be considered as a specific one-dimensional version of a Cosserat continuum. Reddy [26] reformulated various beam theories such as Euler–Bernoulli,

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Timoshenko, Reddy, and Levinson beam theory using Eringen's nonlocal differential constitutive model. The analytical solutions for bending, buckling, and free vibrations were also presented in [26]. Various shear deformation beam theories were also reformulated in recent works by Reddy [35] using nonlocal differential constitutive relations. Similar works have been carried out to study bending, buckling, and free vibration of nanobeams by Aydogdu [27], Civalek [28].

Classical continuum mechanics takes exclusively the bulk into account, nevertheless, neglecting possible contributions from the surface of the deformable body. However, surface effects play a crucial role in the material behavior, the most prominent example being surface tension . A mathematical framework was first developed by Gurtin [36] to study the mechanical behavior of material surfaces. The effect of surface stress on wave propagation in solids has also been studied by Gurtin [37]. The tensorial nature of surface stress was established using the force and moment balance laws. Bodies whose boundaries are material surfaces are discussed and the relation between surface and body stress examined in a recent work by Steinmann [38] and by Hamilton [39]. The surface effects has been applied to modeling two- [40] and three-dimensional continua in the frame work of finite element method [41], [42]. Similar studies on static analysis of nano beams using nonlocal finite element models have been conducted by Mahmoud [43].

Eringen's nonlocal elasticity theory has also been applied to bending, buckling, and vibration of nano beams using the Timoshenko beam theory (see [44], [45], [46] and [47]). Analytical solutions for beam bending problems for different boundary conditions were derived using the nonlocal elasticity theory and the Timoshenko beam theory by Wang et al. [48]. A finite element framework for nonlocal analysis of beams has also been made in a recent work by Sciarra et al. [49]. Nonlocal elastic rod models have been developed to investigate the small-scale effect on axial vibrations of the nano rods by Aydogdu [50] and Adhikari et al[51]. Free vibration analysis of microtubules based on nonlocal theory and the Euler–Bernoulli beam theory was carried out by Civalek et al. [28]. Free vibration analysis of functionally graded carbon nanotubes with various thicknesses, based on the Timoshenko beam theory, has been investigated to obtain numerical solutions us-

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ing Differential Quadrature Method (DQM) by Janghorban et al. [52] and others [53],[54], [55]. Analytical study on the nonlinear free vibration of functionally graded nano beams incorporating surface effects has been presented in [56], [57] and [58]. The effect of surface stresses on bending properties of metal nanowires is presented in [59]. Free vibration analysis of rotating nano cantilevers using non-local theory and the Euler–Bernoulli beam theory has been carried out in [60] and [61].

The focus of this work is on nonlocal nonlinear formulation together with surface effects for static and free vibration analysis of rotating layered nano cantilever beams using Timoshenko beam theory. A non-local nonlinear finite element formulation for the case of nonuniform rotating isotropic and laminated nano cantilever beams using the Timoshenko beam theory is presented. The surface stress effects and the geometric nonlinearity is taken into account by using the Green Lagrange strain tensor. Numerical results are presented to bring out the parametric effects of nonlocal parameter and surface stresses on bending and vibration behavior of layered nano cantilever beams.

2 Nonlocal Theories

In classical elasticity, stress at a point is a function of strain at that point. Whereas in nonlocal elasticity, stress at a point is a function of strains at all points in the continuum. In nonlocal theories, forces between the atoms and internal length scale are considered in the constitutive equation. Nonlocal theory was first introduced by Eringen [3]. According to Eringen, the stress field at a point x in an elastic continuum not only depends on the strain field at that point but also on the strains at all other points of the body. Eringen attributed this fact to the atomic theory of lattice dynamics and experimental observation on phonon dispersion. The nonlocal stress tensor σ at a point x in the continuum is expressed as Eringen expressed a constitutive model that expresses

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the nonlocal stress tensor at a point x as

$$\sigma_{ij} = \int_v K(|x' - x|, \tau) t_{ij}(x') dv(x') \tag{1}$$

where the volume intergarl in Eq. (1) is over the region v occupied by the body and t_{ij} is the Hookean stress stress tensor at point x defined as

$$t_{ij} = c_{ijkl}\varepsilon_{kl} \tag{2}$$

and the kernel function $K(|x' - x|, \tau)$ represents the nonlocal modulus, $|x' - x|$ is the distance and τ is the material constant that depends on internal and external characteristic lengths. The Kernel function can be obtained by matching the lattice dynamics with nonlocal results [24]. For example Kernel function for 2-D problems has the form

$$K(|x|) = (2\pi^2 l^2 \tau^2)^{-1} K_0(|x|/l\tau), \tau = e_0 a/l, \tag{3}$$

where K_0 is the modified Bessel function, a and l are internal and external characteristic lengths, e_0 is the material constant which is defined by the experiment. In the nonlocal linear elasticity, equations of motion can be obtained from nonlocal balance law

$$\sigma_{ij,j} + f_i = \rho\ddot{u} \tag{4}$$

where i, j take symbols x, y, z and f_i, ρ and u_i are the components of the body force, mass density and displacement vector. By substituting Eq. (1) into Eq. (4), the integral form of nonlocal constitutive equation is obtained. Because solving an integral equation is more difficult than a differential equation, Eringen [24] proposed a differential form of nonlocal constitutive equation as

$$(1 - \tau^2 l^2 \nabla^2)\sigma_{ij} = t_{ij} \tag{5}$$

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Eq. (5) is more convenient than the integral relation (1) to apply to various linear elasticity problems.

3 Laminated Beams

3.1 Classical Timoshenko Beam Theory

In the Timoshenko beam theory, the effects of shear deformation is also considered. Distribution of transverse shear stress is assumed to be constant through the thickness. The displacement field is given by

$$\mathbf{u}(x,z,t) = \left[u(x,t) + z\varphi_x\right]\hat{\mathbf{e}}_x + w(x,t)\hat{\mathbf{e}}_z \tag{6}$$

The non-zero components of Lagrangian strain tensor can be written as

$$\begin{aligned}\varepsilon_{xx} &= \left[\frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2\right] + z\left[\frac{d\varphi_x}{dx}\right]\\ &= \left[\varepsilon_{xx}^{(0)}\right] + z\left[\varepsilon_{xx}^{(1)}\right] &\text{(7a)}\\ \varepsilon_{xz} &= \frac{1}{2}\left(\varphi_x + \frac{dw}{dx}\right) &\text{(7b)}\\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}[(\frac{\partial u}{\partial z})^2 + (\frac{\partial w}{\partial z})^2] = \frac{1}{2}\varphi_x^2 &\text{(7c)}\end{aligned}$$

and ε_{zz} is positive and non-zero. Its contribution is through a material length scale. Therefore, for a laminate layer, following stress-strain relationship is used

$$\begin{aligned}\sigma_{xx} &= \bar{Q}_{11}(\varepsilon_{xx} - \varsigma_{xx}\Delta T) + \alpha\varepsilon_{zz}\\ \sigma_{xz} &= K_s\bar{Q}_{55}\gamma_{xz} &\text{(8a)}\\ \sigma_{zz} &= \alpha\varepsilon_{xx} + \beta\varepsilon_{zz}\\ \sigma^s &= \tau^0 + E^s\varepsilon_{xx} &\text{(8b)}\end{aligned}$$

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where, α and β are material length scale parameters, K_s is shear correction factor and

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta \tag{9}$$

$$\bar{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta \tag{10}$$

$$\varsigma_{xx} = \varsigma_1\cos^2\theta + \varsigma_2\sin^2\theta \tag{11}$$

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, Q_{66} = G_{12} \tag{12}$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{55} = G_{13}, \quad Q_{44} = G_{23} \tag{13}$$

where, E_1, E_2, ν_{12}, G_{12}, G_{13} and G_{23} are six independent engineering constants and θ is the orientation of the laminate layer. The axial force due to rotation of a cantilever beam is given as

$$N_{ax} = \frac{\rho A\Omega^2}{2}(L-x)(L+x-2r) \tag{14}$$

Therefore, the stress resultants can be written as

$$N_{xx} = \sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} b\sigma_{xx}\,dz + \oint_\Gamma \sigma^s\,ds + +\frac{\rho A\Omega^2}{2}(L-x)(L+x-2r) \tag{15a}$$

$$M_{xx} = \sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} b\sigma_{xx}z\,dz + \oint_\Gamma \sigma^s z\,ds \tag{15b}$$

$$N_{zz} = \sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} b\sigma_{zz}\,dz \tag{15c}$$

$$N_{xz} = \sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} b\sigma_{xz}\,dz \tag{15d}$$

Using equations (7) and (8), the stress resultants in equation (15) can be written as

$$N_{xx} = \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{B}\varepsilon_{xx}^{(1)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \tilde{H}\Delta T + \frac{\rho A\Omega^2}{2}(L-x)(L+x-2r) \tag{16a}$$

$$M_{xx} = \tilde{B}\varepsilon_{xx}^{(0)} + \tilde{D}\varepsilon_{xx}^{(1)} + \tilde{J}\varepsilon_{zz} - \tilde{O}\Delta T \tag{16b}$$

$$N_{zz} = \tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \tag{16c}$$

$$N_{xz} = \tilde{G}\gamma_{xz} \tag{16d}$$

where

$$\tilde{A} = \sum_{k=1}^{N} \bar{Q}_{11}^{(k)}(z_{k+1} - z_k)b + 2E^s\,(b+h)\,, \quad \tilde{B} = \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{11}^{(k)}(z_{k+1}^2 - z_k^2)b \tag{17a}$$

$$\tilde{D} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{11}^{(k)}(z_{k+1}^3 - z_k^3)b + E^s\left[\frac{h^3}{6} + \frac{bh^2}{2}\right], \quad \tilde{G} = K_s\sum_{k=1}^{N} \bar{Q}_{55}(z_{k+1} - z_k)b \tag{17b}$$

$$\tilde{C} = \sum_{k=1}^{N} \alpha(z_{k+1} - z_k)b, \quad \tilde{F} = \sum_{k=1}^{N} \beta(z_{k+1} - z_k)b, \quad \tilde{H} = \sum_{k=1}^{N} \bar{Q}_{11}^{(k)}\varsigma_{xx}^{(k)}(z_{k+1} - z_k)b \tag{17c}$$

$$\tilde{J} = \frac{1}{2}\sum_{k=1}^{N}(z_{k+1}^2 - z_k^2)b, \quad \tilde{O} = \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{11}^{(k)}\varsigma_{xx}^{(k)}(z_{k+1}^2 - z_k^2)b \tag{17d}$$

3.2 Nonlocal Timoshenko Beam Theory

Using equation (5), the nonlocal stress resultants in terms of strains can be written as

$$N_{xx}^{nl} = \mu\frac{d^2N_{xx}^{nl}}{dx^2} + \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{B}\varepsilon_{xx}^{(1)} + \tilde{C}\varepsilon_{zz} + 2\tau^0\,(b+h) - \tilde{H}\Delta T + \frac{\rho A\Omega^2}{2}(L-x)(L+x-2r) \tag{18}$$

$$M_{xx}^{nl} = \mu\frac{d^2M_{xx}^{nl}}{dx^2} + \tilde{B}\varepsilon_{xx}^{(0)} + \tilde{D}\varepsilon_{xx}^{(1)} + \tilde{J}\varepsilon_{zz} - \tilde{O}\Delta T \tag{19}$$

$$N_{zz}^{nl} = \tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \tag{20}$$

$$N_{xz}^{nl} = \mu\frac{d^2N_{xz}^{nl}}{dx^2} + \tilde{G}\gamma_{xz} \tag{21}$$

3.3 Equations of motion

Let us consider the transversely applied point load on the cantilever beam as dirac delta function given as

$$P = Q_0\delta(x - x_p) \tag{22}$$

where Q_0 is the point load applied at the point x_p on the beam. By using the principle of virtual work, the equations of motion for cantilever beam using Timoshenko beam theory can be obtained as

$$\frac{dN_{xx}}{dx} + f_x = m_0 \frac{d^2u}{dt^2} \tag{23}$$

$$\frac{d}{dx}\left[N_{xz} + N_{xx}\frac{dw}{dx}\right] + f_z + P = m_o \frac{d^2w}{dt^2} \tag{24}$$

$$\frac{dM_{xx}}{dx} - (N_{xz} + N_{zz}\varphi_x) = m_1 \frac{d^2\varphi}{dt^2} \tag{25}$$

where

$$m_i = \int_A \rho z^i \, dA$$

and, f_x and f_z are the axially and transversely distributed forces, respectively. Manipulating the equations of motion and using equations (18) to (21), the following relations are obtained:

$$\begin{aligned} N^{nl}_{xx} &= \tilde{A}\varepsilon^{(0)}_{xx} + \tilde{B}\varepsilon^{(1)}_{xx} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+h) - \mu\frac{df_x}{dx} + \mu m_0 \frac{d^3u}{dxdt^2} \\ &\quad - \tilde{H}\Delta T + \frac{\rho A \Omega^2}{2}(L-x)(L+x-2r) \end{aligned} \tag{26a}$$

$$\begin{aligned} M^{nl}_{xx} &= \tilde{B}\varepsilon^{(0)}_{xx} + \tilde{D}\varepsilon^{(1)}_{xx} + \tilde{J}\varepsilon_{zz} - \tilde{O}\Delta T + \mu m_1 \frac{d^3\varphi}{dxdt^2} + \mu m_0 \frac{d^2w}{dt^2} - \mu f_z - \mu P \\ &\quad - \mu\frac{d}{dx}\left[\left(\tilde{A}\varepsilon^{(0)}_{xx} + \tilde{B}\varepsilon^{(1)}_{xx} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+h) - \mu\frac{df_x}{dx} + \mu m_0 \frac{d^3u}{dxdt^2}\right)\frac{dw}{dx}\right] \\ &\quad - \mu\frac{d}{dx}\left[\left(-\tilde{H}\Delta T + \frac{\rho A\Omega^2}{2}(L-x)(L+x-2r)\right)\frac{dw}{dx}\right] \\ &\quad + \mu\frac{d}{dx}\left[\left(\tilde{C}\varepsilon^{(0)}_{xx} + \tilde{F}\varepsilon_{zz}\right)\varphi_x\right] \end{aligned} \tag{26b}$$

$$N^{nl}_{zz} = \tilde{C}\varepsilon^{(0)}_{xx} + \tilde{F}\varepsilon_{zz} \tag{26c}$$

$$\begin{aligned} N^{nl}_{xz} &= \tilde{G}\gamma_{xz} - \mu\frac{df_z}{dx} \\ &\quad - \mu\frac{d^2}{dx^2}\left[\left(\tilde{A}\varepsilon^{(0)}_{xx} + \tilde{B}\varepsilon^{(1)}_{xx} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+h) - \mu\frac{df_x}{dx} + \mu m_0 \frac{d^3u}{dxdt^2}\right)\frac{dw}{dx}\right] \\ &\quad - \mu\frac{d^2}{dx^2}\left[\left(-\tilde{H}\Delta T + \frac{\rho A\Omega^2}{2}(L-x)(L+x-2r)\right)\frac{dw}{dx}\right] \end{aligned} \tag{26d}$$

By substituting the expressions for nonlocal stress resultants (26) back in the equations of motion (23) to (25), we obtain the equilibrium equation for nonlocal Timoshenko beam theory including surface stress effects as

$$
\begin{aligned}
m_0 \frac{d^2 u}{dt^2} - \mu m_0 \frac{d^4 u}{dx^2 dt^2} &= \frac{d}{dx}\left(\tilde{A}\varepsilon_{xx}^{(0)} + \tilde{B}\varepsilon_{xx}^{(1)} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+h) - \mu \frac{df_x}{dx}\right) \\
&+ \frac{d}{dx}\left(-\tilde{H}\Delta T + \frac{\rho A \Omega^2}{2}(L-x)(L+x-2r)\right) + f_x
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
m_0 \frac{d^2 w}{dt^2} - \mu m_0 \frac{d^4 w}{dx^2 dt^2} &= \frac{d}{dx}\left(\tilde{G}\gamma_{xz}\right) - \mu \frac{d^2 f_z}{dx^2} + f_z + P \\
&+ \frac{d}{dx}\left[\left(\tilde{A}\varepsilon_{xx}^{(0)} + \tilde{B}\varepsilon_{xx}^{(1)} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2}\right)\frac{dw}{dx}\right] \\
&+ \frac{d}{dx}\left[\left(-\tilde{H}\Delta T + \frac{\rho A \Omega^2}{2}(L-x)(L+x-2r)\right)\frac{dw}{dx}\right] \\
&- \mu \frac{d^3}{dx^3}\left[\left(\tilde{A}\varepsilon_{xx}^{(0)} + \tilde{B}\varepsilon_{xx}^{(1)} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+H) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2}\right)\frac{dw}{dx}\right] \\
&- \mu \frac{d^3}{dx^3}\left[\left(-\tilde{H}\Delta T + \frac{\rho A \Omega^2}{2}(L-x)(L+x-2r)\right)\frac{dw}{dx}\right]
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
m_1 \frac{d^2 \varphi_x}{dt^2} - \mu m_1 \frac{d^4 \varphi_x}{dx^2 dt^2} &= \frac{d}{dx}\left(\tilde{B}\varepsilon_{xx}^{(0)} + \tilde{D}\varepsilon_{xx}^{(1)} + \tilde{J}\varepsilon_{zz} - \tilde{O}\Delta T\right) - \tilde{G}\gamma_{xz} - \varphi_x\left(\tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz}\right) \\
&+ \mu \frac{d^2}{dx^2}\left[\left(\tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz}\right)\varphi_x\right]
\end{aligned} \tag{29}
$$

3.4 Finite Element Formulation

The principle of virtual work for the Timoshenko beam has the form

$$
\begin{aligned}
0 = \int_0^l &[N_{xx}^{nl}\delta\varepsilon_{xx}^{(0)} + M_{xx}^{nl}\delta\varepsilon_{xx}^{(1)} + N_{xz}^{nl}\delta\gamma_{xz} + N_{zz}^{nl}\delta\varepsilon_{zz} \\
&- f_x \delta u - f_z \delta w - P\delta w + m_0 \ddot{u}\delta u + m_0 \ddot{w}\delta w + m_1 \ddot{\varphi}\delta\varphi]\, dx \\
&- Q_1 \delta u(0) - Q_4 \delta u(l) - Q_2 \delta w(0) - Q_5 \delta w(l) - Q_3 \delta\varphi(0) - Q_6 \delta\varphi(0)
\end{aligned} \tag{30}
$$

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After substituting the expressions for stress resultants from equation (26) into the equation (30), we obtain

$$\begin{aligned}
0 = \int_0^T \int_0^l \Bigg(& \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{B}\varepsilon_{xx}^{(1)} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+H) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dxdt^2} - \tilde{H}\Delta T \\
& + \frac{\rho A \Omega^2}{2}(L-x)(L+x-2r) \Bigg) \delta\varepsilon_{xx}^{(0)} + \left(\mu m_1 \frac{d^3 \varphi}{dxdt^2} + \mu m_0 \frac{d^2 w}{dt^2} - \mu f_z - \mu P \right) \delta\varepsilon_{xx}^{(1)} \\
& + \left(\tilde{B}\varepsilon_{xx}^{(0)} + \tilde{D}\varepsilon_{xx}^{(1)} + \tilde{J}\varepsilon_{zz} - \tilde{O}\Delta T \right) \delta\varepsilon_{xx}^{(1)} \\
& \underline{-\mu \frac{d}{dx}\left[\left(\tilde{A}\varepsilon_{xx}^{(0)} + \tilde{B}\varepsilon_{xx}^{(1)} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+H) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dxdt^2} \right) \frac{dw}{dx} \right] \delta\varepsilon_{xx}^{(1)}} \\
& \underline{-\mu \frac{d}{dx}\left[\left(-\tilde{H}\Delta T + \frac{\rho A \Omega^2}{2}(L-x)(L+x-2r) \right) \frac{dw}{dx} \right] \delta\varepsilon_{xx}^{(1)}} \\
& + \mu \frac{d}{dx}\left[\left(\tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \right) \varphi_x \right] \delta\varepsilon_{xx}^{(1)} + \left(\tilde{G}\gamma_{xz} + \mu m_0 \frac{d^3 w}{dxdt^2} - \mu \frac{df_z}{dx} \right) \delta\gamma_{xz} \\
& \underline{-\mu \frac{d^2}{dx^2}\left[\left(\tilde{A}\varepsilon_{xx}^{(0)} + \tilde{B}\varepsilon_{xx}^{(1)} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+H) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dxdt^2} \right) \frac{dw}{dx} \right] \delta\gamma_{xz}} \\
& \underline{-\mu \frac{d^2}{dx^2}\left[\left(-\tilde{H}\Delta T + \frac{\rho A \Omega^2}{2}(L-x)(L+x-2r) \right) \frac{dw}{dx} \right] \delta\gamma_{xz}} \\
& + \left(\tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \right) \delta\varepsilon_{zz} - f_x \delta u - f_z \delta w - P\delta w \\
& + m_0 \ddot{u}\delta u + m_1 \ddot{w}\delta w + m_1 \ddot{\varphi}_x \delta\varphi_x \, dx \\
& [-Q_1 \delta u(x_a) - Q_4 \delta u(x_b) - Q_2 \delta w(x_a) - Q_5 \delta w(x_b) - Q_3 \delta\varphi(x_a) - Q_6 \delta\varphi(x_b)] \; dT \qquad (31)
\end{aligned}$$

The underlined expressions in the above equation does not allow us to construct a quadratic functional. So after omitting the underlined expressions in the equation (31), it can be equivalently written into the following three equations

$$\begin{aligned}
\int_0^T \int_0^l \Bigg[& \left(\tilde{A}\varepsilon_{xx}^{(0)} + \tilde{B}\varepsilon_{xx}^{(1)} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+H) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dxdt^2} \right) \frac{d\delta u}{dx} \\
& + \left(-\tilde{H}\Delta T + \frac{\rho A \Omega^2}{2}(L-x)(L+x-2r) \right) \frac{d\delta u}{dx} - f_x \delta u + m_0 \ddot{u}\delta u \, dx \\
& + [-Q_1 \delta u(x_a) - Q_4 \delta u(x_b)] \; dT \Bigg] \qquad (32)
\end{aligned}$$

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$$\int_0^T \int_0^l \left(\tilde{A}\varepsilon_{xx}^{(0)} + \tilde{B}\varepsilon_{xx}^{(1)} + \tilde{C}\varepsilon_{zz} + 2\tau^0 (b+H) - \mu\frac{df_x}{dx} + \mu m_0 \frac{d^3u}{dxdt^2}\right)\frac{dw}{dx}\frac{d\delta w}{dx}$$
$$+ \left(-\tilde{H}\Delta T + \frac{\rho A\Omega^2}{2}(L-x)(L+x-2r)\right)\frac{dw}{dx}\frac{d\delta w}{dx}$$
$$+ \left(\tilde{G}\gamma_{xz} + \mu m_0\frac{d^3w}{dxdt^2} - \mu\frac{df_z}{dx}\right)\frac{d\delta w}{dx} - f_z\delta w - P\delta w + m_0\ddot{w}\delta w\, dx$$
$$+ \left[Q_2\delta w(x_a) - Q_5\delta w(x_b)\right] dT \quad = \quad 0 \tag{33}$$

$$\int_0^T \int_0^l \left(\tilde{G}\gamma_{xz} + \mu m_0\frac{d^3w}{dxdt^2} - \mu\frac{df_z}{dx}\right)\delta\varphi_x$$
$$+ \left(\mu m_1\frac{d^3\varphi_x}{dxdt^2} + \mu m_0\frac{d^2w}{dt^2} - \mu f_z - \mu P\right)\frac{d\delta\varphi_x}{dx}$$
$$+ \left(\tilde{B}\varepsilon_{xx}^{(0)} + \tilde{D}\varepsilon_{xx}^{(1)} + \tilde{J}\varepsilon_{zz} - \tilde{O}\Delta T\right)\frac{d\delta\varphi_x}{dx}$$
$$+ \left(\tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz}\right)\varphi_x\delta\varphi_x + m_1\ddot{\varphi}_x\delta\varphi_x\, dx$$
$$+ \left[-Q_3\delta\varphi_x(x_a) - Q_6\delta\varphi_x(x_b)\right] dT \quad = \quad 0 \tag{34}$$

The generalized displacements $(\bar{u}, \bar{w}, \bar{\varphi}_x)$ are approximated using the Lagrange interpolation functions

$$\bar{u}(x) = \sum_{j=1}^{m} \Delta_j^1 \psi_j^{(1)}(x) \tag{35a}$$
$$\bar{w}(x) = \sum_{j=1}^{n} \Delta_j^2 \psi_j^{(2)}(x) \tag{35b}$$
$$\bar{\varphi}_x(x) = \sum_{j=1}^{p} \Delta_j^3 \psi_j^{(3)}(x) \tag{35c}$$

By subsituting equation (35) for $\bar{u}$, $\bar{w}$ and $\bar{\varphi}_x$, and putting $\delta\bar{u} = \psi_i^1$, $\delta\bar{w} = \psi_i^2$, $\delta\bar{\varphi}_x = \psi_i^3$ into the weak form statements (32)-(34), the finite element model of the Timoshenko beam can be expressed as

$$\begin{bmatrix} K^{11} & K^{12} & K^{13} \\ K^{21} & K^{22} & K^{23} \\ K^{31} & K^{32} & K^{33} \end{bmatrix} \begin{Bmatrix} \Delta^1 \\ \Delta^2 \\ \Delta^3 \end{Bmatrix} + \begin{bmatrix} M^{11} & M^{12} & M^{13} \\ M^{21} & M^{22} & M^{23} \\ M^{31} & M^{32} & M^{33} \end{bmatrix} \begin{Bmatrix} \ddot{\Delta}^1 \\ \ddot{\Delta}^2 \\ \ddot{\Delta}^3 \end{Bmatrix} = \begin{Bmatrix} F^1 \\ F^2 \\ F^3 \end{Bmatrix} \tag{36}$$

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where the stiffness coefficients $K_{ij}^{\alpha\beta}$, mass matrix coefficients $M_{ij}^{\alpha\beta}$ and force coefficients F_i^{α} $(\alpha, \beta = 1, 2, 3)$ are defined as follows:

$$
\begin{aligned}
K_{ij}^{11} &= \int_0^l \tilde{A}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(1)}}{dx}\,dx \\
K_{ij}^{12} &= \int_0^l \frac{1}{2}\tilde{A}\frac{dw}{dx}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(2)}}{dx}\,dx \\
K_{ij}^{13} &= \int_0^l \frac{1}{2}\tilde{C}\varphi_x\frac{d\psi_i^{(1)}}{dx}\psi_j^{(3)}\,dx + \int_0^l \tilde{B}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(3)}}{dx}\,dx \\
K_{ij}^{21} &= \int_0^l \tilde{A}\frac{dw}{dx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(1)}}{dx}\,dx \\
K_{ij}^{22} &= \int_0^l \tilde{G}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx} \\
&\qquad + \frac{1}{2}\left\{\tilde{A}\left(\frac{dw}{dx}\right)^2 + \tilde{C}\,(\varphi_x)^2\right\}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}\,dx \qquad (37) \\
K_{ij}^{23} &= \int_0^l \tilde{G}\frac{d\psi_i^{(2)}}{dx}\psi_j^{(3)}\,dx + \int_0^l \tilde{B}\frac{dw}{dx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(3)}}{dx} \\
K_{ij}^{31} &= \int_0^l \tilde{C}\varphi_x\psi_i^{(3)}\frac{d\psi_j^{(1)}}{dx}\,dx + \int_0^l \tilde{B}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(1)}}{dx}\,dx \\
K_{ij}^{32} &= \int_0^l \tilde{G}\psi_i^{(3)}\frac{d\psi_j^{(2)}}{dx}\,dx + \int_0^l \frac{1}{2}\tilde{B}\frac{dw}{dx}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(2)}}{dx}\,dx \\
K_{ij}^{33} &= \int_0^l \tilde{D}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(3)}}{dx} + \tilde{G}\psi_i^{(3)}\psi_j^{(3)} + \frac{1}{2}\tilde{J}\varphi_x\frac{d\psi_i^{(3)}}{dx}\psi_j^{(3)} \\
&\qquad + \frac{1}{2}\left\{\tilde{C}\left(\frac{dw}{dx}\right)^2 + \tilde{F}(\varphi_x)^2\right\}\psi_i^{(3)}\psi_j^{(3)}\,dx
\end{aligned}
$$

$$
\begin{aligned}
M_{ij}^{11} &= m_0\psi_i^{(1)}\psi_j^{(1)} + \mu m_0\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(1)}}{dx} \\
M_{ij}^{21} &= \mu m_0\frac{dw}{dx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(1)}}{dx} \\
M_{ij}^{22} &= m_0\psi_i^{(2)}\psi_j^{(2)} + \mu m_0\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx} \qquad (38)
\end{aligned}
$$

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$$
\begin{aligned}
M_{ij}^{32} &= \mu m_0 \psi_i^{(3)} \frac{d\psi_j^{(2)}}{dx} + \mu m_0 \frac{d\psi_i^{(3)}}{dx} \psi_j^{(2)} \\
M_{ij}^{33} &= \mu m_1 \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} + m_1 \psi_i^{(3)} \psi_j^{(3)} \\
M_{ij}^{12} &= 0 \quad , \quad M_{ij}^{13} = 0 \quad , \quad M_{ij}^{23} = 0 \quad , \quad M_{ij}^{31} = 0 \\
F_i^1 &= \int_0^l \left\{ f_x \psi_i^{(1)} + \mu \frac{df_x}{dx} \frac{d\psi_i^{(1)}}{dx} - 2\tau^0 (b+h) \frac{d\psi_i^{(1)}}{dx} \right\} dx \\
&\quad + \int_0^l \left(\tilde{H} \Delta T - \frac{\rho A \Omega^2}{2} (L-x)(L+x-2r) \right) \frac{d\psi_i^{(1)}}{dx} dx \\
&\qquad + Q_1 \psi_i^{(1)}(0) + Q_4 \psi_i^{(1)}(l) \\
F_i^2 &= \int_0^l \left\{ f_z \psi_i^{(2)} + P \psi_i^{(2)} + \mu \left(\frac{df_z}{dx} + \frac{df_x}{dx} \frac{dw}{dx} \right) \frac{d\psi_i^{(2)}}{dx} - 2\tau^0 (b+h) \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \right\} dx \\
&\quad + \int_0^l \left(\tilde{H} \Delta T - \frac{\rho A \Omega^2}{2} (L-x)(L+x-2r) \right) \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} dx \\
&\quad + Q_2 \psi_i^{(2)}(0) + Q_5 \psi_i^{(2)}(l) \\
F_i^3 &= \int_0^l \left[\mu \left(f_z \frac{d\psi_i^{(3)}}{dx} + P \frac{d\psi_i^{(3)}}{dx} + \frac{df_z}{dx} \psi_i^{(3)} \right) + \tilde{O} \Delta T \frac{d\psi_i^{(3)}}{dx} \right] dx + Q_3 \psi_i^{(3)}(0) + Q_6 \psi_i^{(3)}(l)
\end{aligned}
\tag{39}
$$

4 Reduction to the case of Isotropic Beams

The displacement field and corresponding strains are same as given in (6) and (7). For an isotropic material, following stress-strain relationship is used:

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$$
\begin{aligned}
\sigma_{xx} &= E\varepsilon_{xx} + \alpha\varepsilon_{zz} - E\varsigma\Delta T\,\varepsilon_{xx} \\
\sigma_{xz} &= GK_s\gamma_{xz} \\
\sigma_{zz} &= \alpha\varepsilon_{xx} + \beta\varepsilon_{zz} \\
\sigma^s &= \tau^0 + E^s\varepsilon_{xx}
\end{aligned} \tag{40}
$$

where α and β are material length scale parameters, and E, G, K_s and ς are Young's moduli, shear moduli, shear correction factor and co-efficient of thermal expansion, respectively. The axial force due to rotation of a cantilever beam is given as

$$N_{ax} = \frac{\rho A\Omega^2}{2}(L - x)(L + x - 2r) \tag{41}$$

where ρ is the mass density, Ω is the angular velocity of rotation and r is the hub radius as shown in Figure 1. The stress resultants can be written as

$$N_{xx} = \int_A \sigma_{xx}\,dA + \oint_\Gamma \sigma^s\,ds + \frac{\rho A\Omega^2}{2}(L - x)(L + x - 2r) \tag{42a}$$

$$M_{xx} = \int_A \sigma_{xx} z\,dA + \oint_\Gamma \sigma^s z\,ds \tag{42b}$$

$$N_{zz} = \int_A \sigma_{zz}\,dA \tag{42c}$$

$$N_{xz} = \int_A \sigma_{xz}\,dA \tag{42d}$$

Using equations (7) and (8), the stress resultants in equation (42) can be written as

$$N_{xx} = \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0\,(b + h) - EA\varsigma\Delta T + \frac{\rho A\Omega^2}{2}(L - x)(L + x - 2r) \tag{43a}$$

$$M_{xx} = \tilde{D}\varepsilon_{xx}^{(1)} \tag{43b}$$

$$N_{zz} = \tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \tag{43c}$$

$$N_{xz} = \tilde{G}\gamma_{xz} \tag{43d}$$

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where

$$\tilde{A} = EA + 2E^s (b + h) \tag{44}$$

$$\tilde{C} = \alpha A \tag{45}$$

$$\tilde{D} = EI + E^s \left[\frac{h^3}{6} + \frac{bh^2}{2} \right] \tag{46}$$

$$\tilde{F} = \beta A \tag{47}$$

$$\tilde{G} = K_s GA \tag{48}$$

The equations of motion of the cantilever beam using Timoshenko beam theory can now be obtained as

$$\frac{dN_{xx}}{dx} + f_x = m_0 \frac{d^2 u}{dt^2} \tag{49}$$

$$\frac{d}{dx} \left[N_{xz} + N_{xx} \frac{dw}{dx} \right] + f_z + P = m_o \frac{d^2 w}{dt^2} \tag{50}$$

$$\frac{dM_{xx}}{dx} - (N_{xz} + N_{zz}\varphi_x) = m_1 \frac{d^2 \varphi}{dt^2} \tag{51}$$

where

$$m_i = \int_A \rho z^i \, dA$$

and f_x and f_z are the axially and transversely distributed forces, respectively.

The generalized displacements $(\bar{u}, \bar{w}, \bar{\varphi}_x)$ are approximated using Lagrange interpolation func-

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tions

$$\bar{u}(x) = \sum_{j=1}^{m} \Delta_j^1 \psi_j^{(1)}(x) \tag{52a}$$

$$\bar{w}(x) = \sum_{j=1}^{n} \Delta_j^2 \psi_j^{(2)}(x) \tag{52b}$$

$$\bar{\varphi}_x(x) = \sum_{j=1}^{p} \Delta_j^3 \psi_j^{(3)}(x) \tag{52c}$$

By subsituting equation (52) for $\bar{u}$, $\bar{w}$ and $\bar{\varphi}_x$, and putting $\delta\bar{u} = \psi_i^1$, $\delta\bar{w} = \psi_i^2$, $\delta\bar{\varphi}_x = \psi_i^3$ into the weak form statements as discussed in previous sections, the finite element model of the Timoshenko beam can be expressed as:

$$\begin{bmatrix} K^{11} & K^{12} & K^{13} \\ K^{21} & K^{22} & K^{23} \\ K^{31} & K^{32} & K^{33} \end{bmatrix} \begin{Bmatrix} \Delta^1 \\ \Delta^2 \\ \Delta^3 \end{Bmatrix} + \begin{bmatrix} M^{11} & M^{12} & M^{13} \\ M^{21} & M^{22} & M^{23} \\ M^{31} & M^{32} & M^{33} \end{bmatrix} \begin{Bmatrix} \ddot{\Delta}^1 \\ \ddot{\Delta}^2 \\ \ddot{\Delta}^3 \end{Bmatrix} = \begin{Bmatrix} F^1 \\ F^2 \\ F^3 \end{Bmatrix} \tag{53}$$

where the stiffness coefficients $K_{ij}^{\alpha\beta}$, mass matrix coefficients $M_{ij}^{\alpha\beta}$ and force coefficients F_i^{α} $(\alpha, \beta =$

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1, 2, 3) are defined as follows:

$$
\begin{aligned}
K_{ij}^{11} &= \int_0^l \tilde{A}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(1)}}{dx}\,dx \\
K_{ij}^{12} &= \int_0^l \frac{1}{2}\tilde{A}\frac{dw}{dx}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(2)}}{dx}\,dx \\
K_{ij}^{13} &= \int_0^l \frac{1}{2}\tilde{C}\varphi_x\frac{d\psi_i^{(1)}}{dx}\psi_j^{(3)}\,dx \\
K_{ij}^{21} &= \int_0^l \tilde{A}\frac{dw}{dx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(1)}}{dx}\,dx \\
K_{ij}^{22} &= \int_0^l \tilde{G}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx} \\
&\qquad + \frac{1}{2}\left\{\tilde{A}\left(\frac{dw}{dx}\right)^2 + \tilde{C}\left(\varphi_x\right)^2\right\}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}\,dx \\
K_{ij}^{23} &= \int_0^l \tilde{G}\frac{d\psi_i^{(2)}}{dx}\psi_j^{(3)}\,dx \\
K_{ij}^{31} &= \int_0^l \tilde{C}\varphi_x\psi_i^{(3)}\frac{d\psi_j^{(1)}}{dx}\,dx \\
K_{ij}^{32} &= \int_0^l \tilde{G}\psi_i^{(3)}\frac{d\psi_j^{(2)}}{dx}\,dx \\
K_{ij}^{33} &= \int_0^l \tilde{D}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(3)}}{dx} + \tilde{G}\psi_i^{(3)}\psi_j^{(3)} \\
&\qquad + \frac{1}{2}\left\{\tilde{C}\left(\frac{dw}{dx}\right)^2 + \tilde{F}(\varphi_x)^2\right\}\psi_i^{(3)}\psi_j^{(3)}\,dx
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
M_{ij}^{11} &= m_0\psi_i^{(1)}\psi_j^{(1)} + \mu m_0\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(1)}}{dx} \\
M_{ij}^{21} &= \mu m_0\frac{dw}{dx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(1)}}{dx} \\
M_{ij}^{22} &= m_0\psi_i^{(2)}\psi_j^{(2)} + \mu m_0\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}
\end{aligned}
\tag{55}
$$

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$$
\begin{aligned}
M_{ij}^{32} &= \mu m_0 \psi_i^{(3)} \frac{d\psi_j^{(2)}}{dx} + \mu m_0 \frac{d\psi_i^{(3)}}{dx} \psi_j^{(2)} \\
M_{ij}^{33} &= \mu m_1 \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} + m_1 \psi_i^{(3)} \psi_j^{(3)} \\
M_{ij}^{12} &= 0 \quad , \quad M_{ij}^{13} = 0 \quad , \quad M_{ij}^{23} = 0 \quad , \quad M_{ij}^{31} = 0 \\
F_i^1 &= \int_0^l \left\{ f_x \psi_i^{(1)} + \mu \frac{df_x}{dx} \frac{d\psi_i^{(1)}}{dx} - 2\tau^0 (b+h) \frac{d\psi_i^{(1)}}{dx} \right\} dx \\
&\quad + \int_0^l \left[EA\varsigma\Delta T - \frac{\rho A \Omega^2}{2} (L-x)(L+x-2r) \right] \frac{d\psi_i^{(1)}}{dx} dx \\
&\quad + Q_1 \psi_i^{(1)}(0) + Q_4 \psi_i^{(1)}(l) \\
F_i^2 &= \int_0^l \left\{ f_z \psi_i^{(2)} + P \psi_i^{(2)} + \mu \left(\frac{df_z}{dx} + \frac{df_x}{dx} \frac{dw}{dx} \right) \frac{d\psi_i^{(2)}}{dx} - 2\tau^0 (b+h) \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \right\} dx \\
&\quad + \int_0^l \left[EA\varsigma\Delta T - \frac{\rho A \Omega^2}{2} (L-x)(L+x-2r) \right] \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} dx \\
&\quad + Q_2 \psi_i^{(2)}(0) + Q_5 \psi_i^{(2)}(l) \\
F_i^3 &= \int_0^l \mu \left[f_z \frac{d\psi_i^{(3)}}{dx} + P \frac{d\psi_i^{(3)}}{dx} + \frac{df_z}{dx} \psi_i^{(3)} \right] dx + Q_3 \psi_i^{(3)}(0) + Q_6 \psi_i^{(3)}(l)
\end{aligned}
\tag{56}
$$

5 Numerical results

We will present numerical examples to demonstrate the application of the above non-linear non-local formulation in this section. Static bending behavior of both isotropic and laminated beam are studied in the first example. Free vibration analysis of both isotropic and laminated beam are carried out in the second example. Clamped free boundary conditions (C-F) are considered in each example. Sinusoidal distribution of load with the intensity q_0 is used. Numerical implementation is made after developing a MATLAB code for the Timoshenko beam finite element as discussed in the previous section.

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For the static bending analysis of the beam, the following cases are considered for the parametric study, namely (a) the effect of non-local parameter μ, (b) the effect of surface modulus E_s and (c) the effect of surface tension parameter τ on the nonlinear behavior of both isotropic and laminated beam. For the free vibration analysis, the following cases are considered for the parametric study, namelely (a) the variation of fundamental frequency ratio with aspect ratio for different values of non-local parameter μ, (b) the effect of surface modulus E_s on the variation of fundamental frequency with aspect ratio of both isotropic and laminated beam.

C-F beam: $u_{(x=0)} = 0, \quad w_{(x=0)} = 0, \quad \phi_{(x=0)} = 0$

5.1 Example 1: Static bending analysis

The material properties of the isotropic beam are taken as: Elastic modulus $E = 17.73 \text{ x } 10^{10}$ N/m^2 and Poisson's ratio $\nu = 0.27$. The material properties of the laminated

beam are taken as: $E_{11} = 140 \text{ x } 10^9$ N/m^2, $E_{22} = 10 \text{ x } 10^9$ N/m^2 and $\nu_{12} = 0.3$. Four layered cross ply laminate (0/90/0/90) is considered. The boundary condition read as: To study the effect of non-local parameter μ on the non-linear behavior of the beam, the non-local parameter μ is varied from 0 to 5 nm^2. The plot of non-local parameter μ versus the central deflection w is shown in Figure 2. It is observed that for both isotropic beam and laminated beam with increase in the non-local parameter μ there is a decrease in the bending behavior.

To study the effect of the surface modulus E_s on the non-local non-linear behavior, the surface modulus values of 0 N/m, 13 N/m and −3 N/m are taken. The plot of intensity of distributed load versus central transverse deflection is shown in Figure 3. For the positive value of surface modulus E_s, there is an increase in stiffness of the beam and hence reduction in the deflection of the beam. Negative values of E_s tends to decrease the stiffness of the beam and hence results in reduced deflection. To study the effect of surface tension τ on the bending behavior of the beam, a surface tension value of $\tau = 1.7$ N/ is taken for analysis. A plot of intensity of distributed load versus

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center deflection for different values of τ is presented in Figure 4. The surface tension τ stiffens the beam and reduces the deflections.

5.2 Example 2: Free vibration analysis

In this example, a beam with aspect ratio L/H varying from 10 to 50 is considered for the non-local non-linear free vibration analysis. The material properties for isotropic and laminated beam are taken same as in the previous example.Clamped-Free (C-F) boundary condition is considered. To study the effect of non-local parameter on the variation of frequency ratio with the aspect ratio of the beam, various nonlocal parameters μ from 0 to 5 nm^2 are taken. The frequency ratio is defined as

$$\text{Frequency ratio} = \frac{\omega_{nl}(\text{with non-local effect})}{\omega_{nl}(\text{without non-local effect})}$$

The plot of frequency ratio versus aspect ratio L/H of isotropic and laminated beams are presented in Figure 5. It is observed that with the increase in the non-local parameter μ there is a decrease in the natural frequency of vibration of the beam. This trend is same for both isotropic beam and laminated beam.

The effect of surface modulus E_s on the non-linear natural frequency versus aspect ratio is studied. The plots for both isotropic and laminated beams are preseted in Figure 6. Positive values of E_s stiffen the beam and thus resulting in higher frequencies. Negative values of E_s have the opposite effect and decrease the frequencies. Surface tension τ has no effect on the vibration characteristics of the beam.

Figure 7 shows the variation of natural frequency with non-dimensional angular velocity λ (see Equation (57))for different values on non-local parameter μ. It is observed that as the dimensionless angular velocity increases, the natural frequency also increases. It is also seen that with increase

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in nonlocality there is a increase in natural frequencies of vibration.

$$\lambda^4 = \frac{\rho A L^4}{EI}\Omega^2 \tag{57}$$

The variation of natural frequency with the variation of hub radius for different non-local parameter values is presented in Figure 8. It is seen that natural frequency decreases with increase in the hub radius. It is also seen that with increase in nonlocality there is a increase in natural frequencies of vibration.

The variation of natural frequency with the variation of dimensionless angular velocity for different surface modulus E_s values is presented in Figure 9. Natural frequencies of vibration are higher for positive value of E_s and lower for negative values of E_s.

6 Summary and Conclusions

The effects of non-local parameter and surface stress on non-linear bending and vibration characteristics of beams are studied using the Timoshenko beam theory and Eringen's non-local differential model together with Gurtin and Murdoch surface elasticity theory. Simplified Green–Lagrange strain tensor is used to model geometric non-linearity. The finite element method is used to solve the resulting non-linear equations. Parametric studies are carried out to investigate the influence of non-local parameter (μ), surface parameters (E_s and τ), hub radius (r) and angular velocity (λ). It is found that nonlocal parameter stiffens the cantilever beam resulting in lower deflections and higher natural frequencies of vibration. Positive values of E_s relaxes the beam stiffness resulting in lower deflections and higher frequencies. It is also found that with increase in angular velocity of rotation the natural frequency increases and with increase in hub radius the natural frequency decreases.

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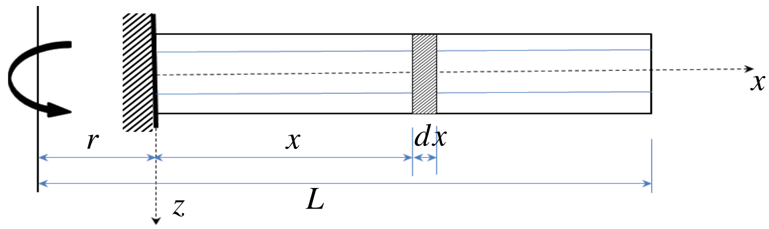

(a)

Figure 1: Rotating nano laminated cantilever beam

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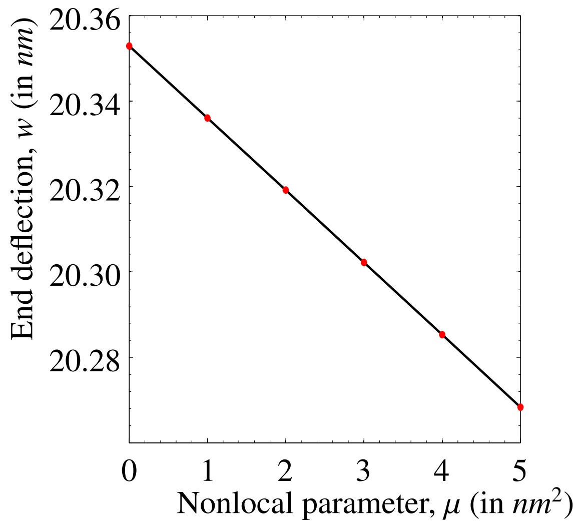

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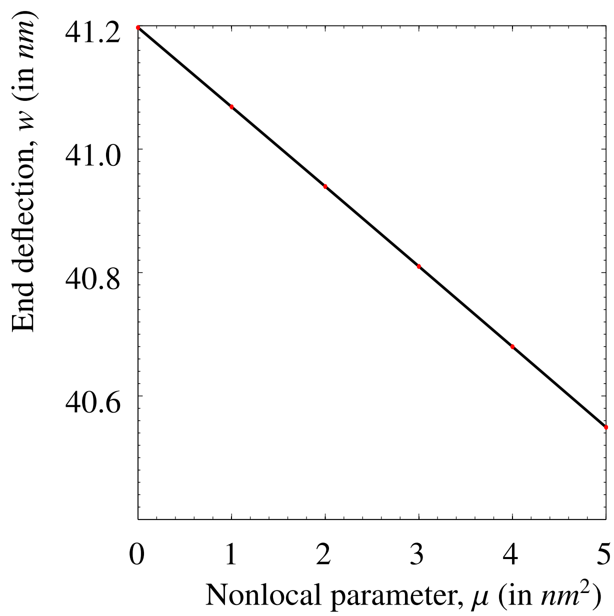

(b)

Figure 2: Plot of nonlocal parameter versus end deflection of the cantilever beam subjected to point load ($Q_0 = 10N$) at the end (for $L/H = 20$) for (a) isotropic beam and (b) laminated beam.

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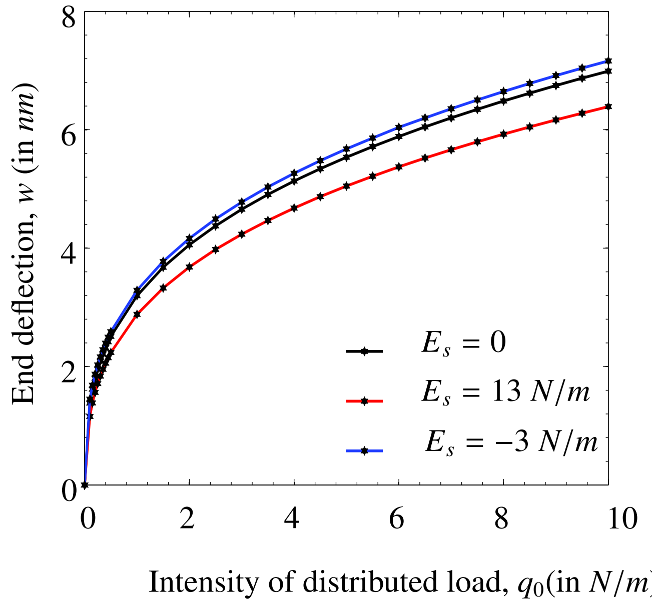

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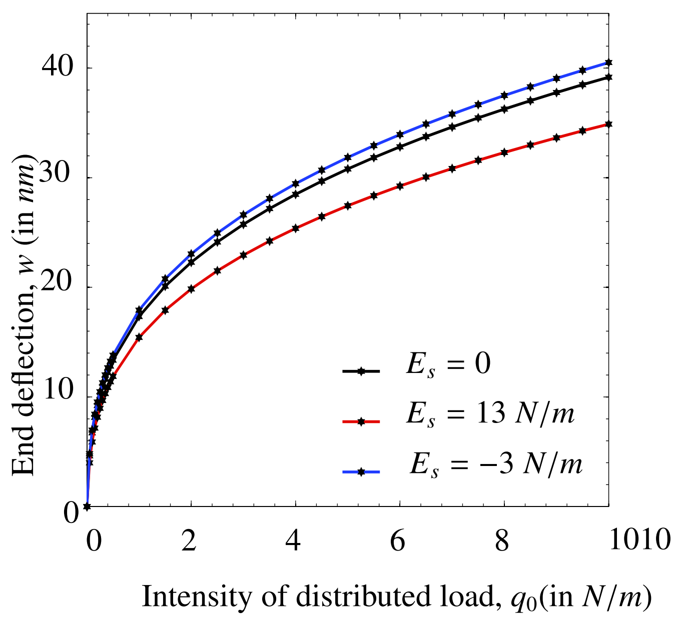

(b)

Figure 3: Plot of load versus end deflection for different values of surface parameter E_s for (a) Isotropic beam and (b) laminated beam.

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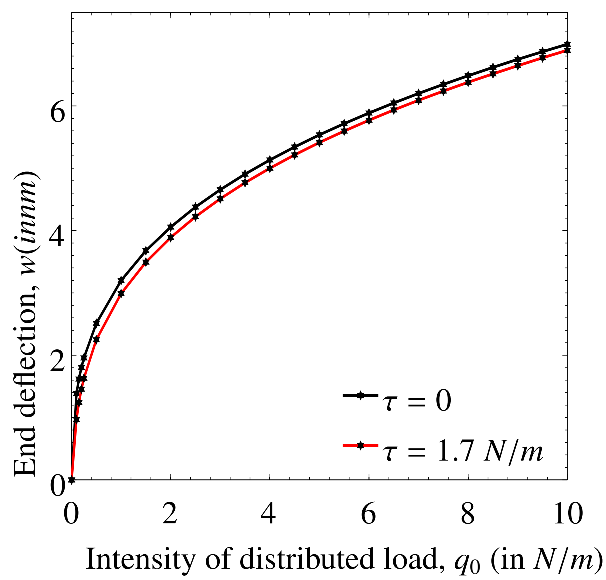

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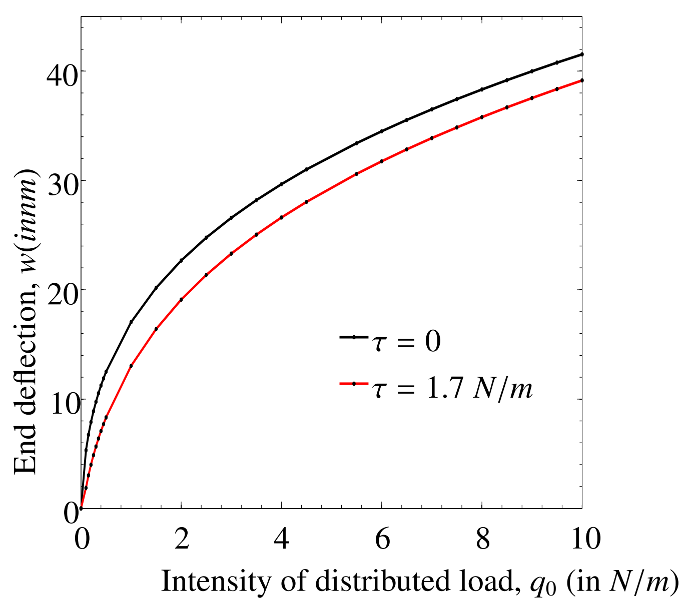

(b)

Figure 4: Plot of load versus end deflection for different values of surface tension parameter τ for (a) Isotropic beam and (b) laminated beam.

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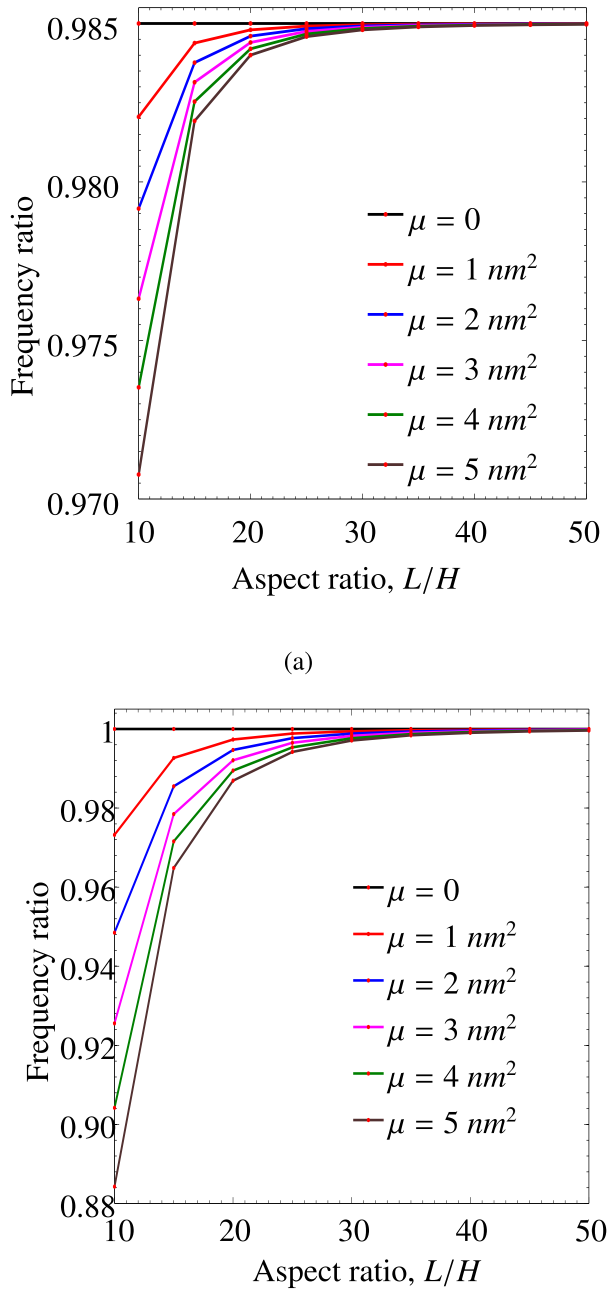

(b)

Figure 5: Plot of aspect ratio L/H versus frequency ratio for different values of nonlocal parameter μ for (a) isotropic beam and (b) laminated beam.

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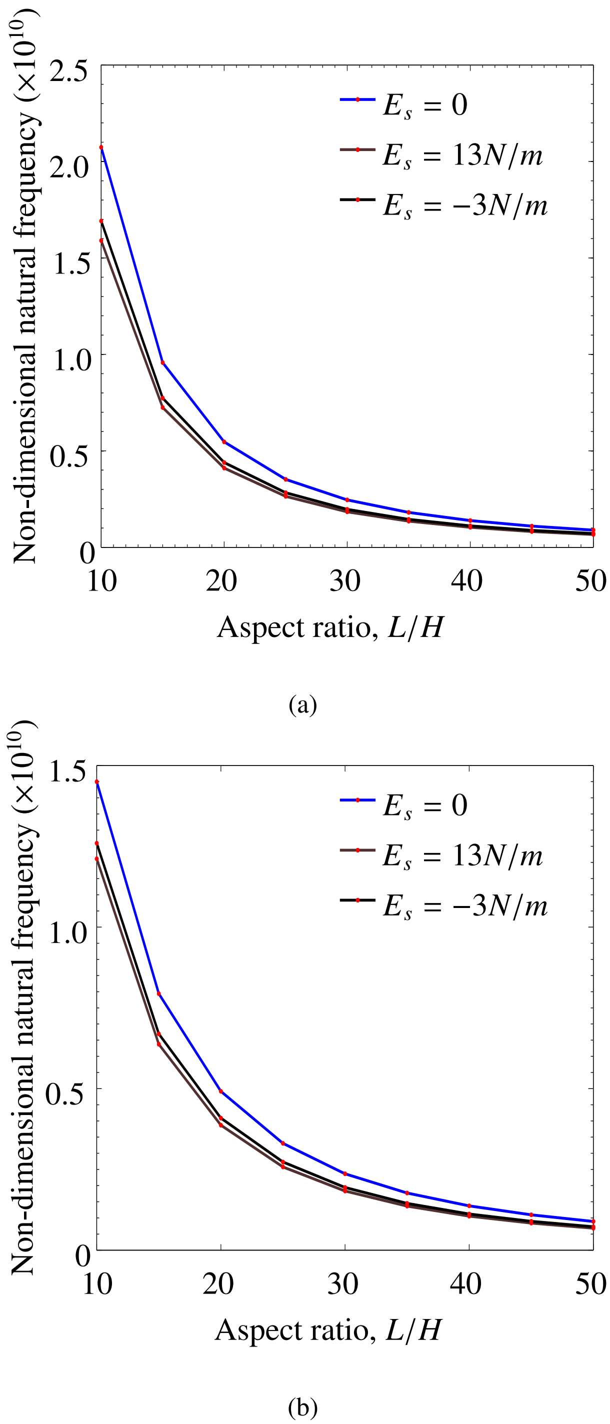

Figure 6: Plot of aspect ratio L/H versus non-dimensional natural frequency for different values of surface modulus E_s for (a) isotropic beam and (b) laminated beam.

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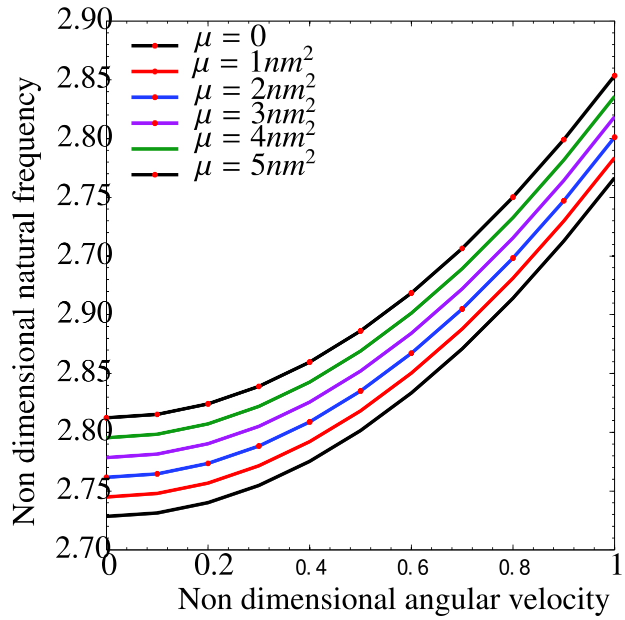

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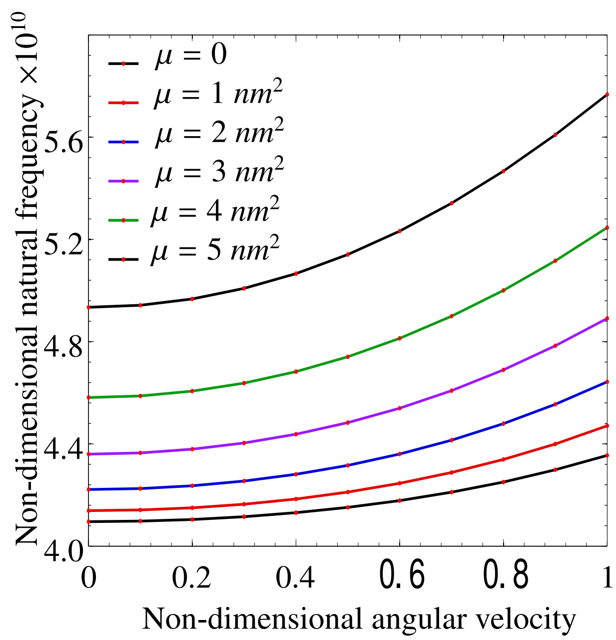

(b)

Figure 7: Plot of non-dimensional angular velocity versus non-dimensional natural frequency for different values of nonlocal parameter for (a) isotropic beam and (b) laminated beam.

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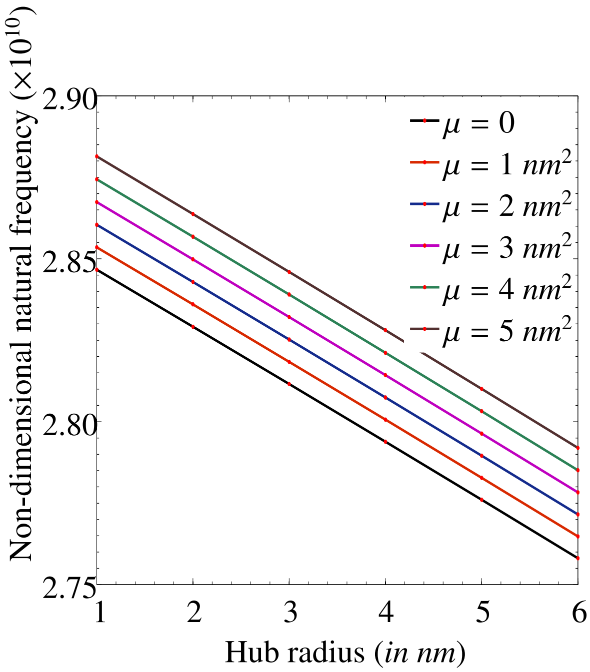

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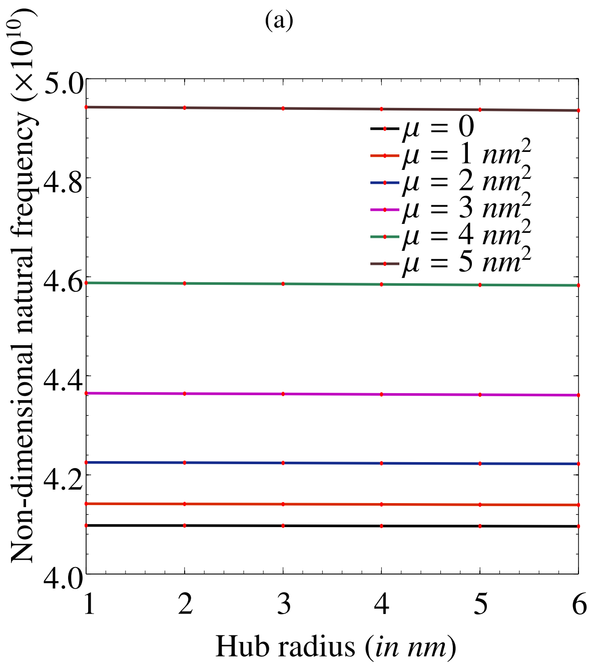

(b)

Figure 8: Plot of hub radius versus non-dimensional natural frequency for different values of nonlocal paramter μ for (a) isotropic beam and (b) laminated beam.

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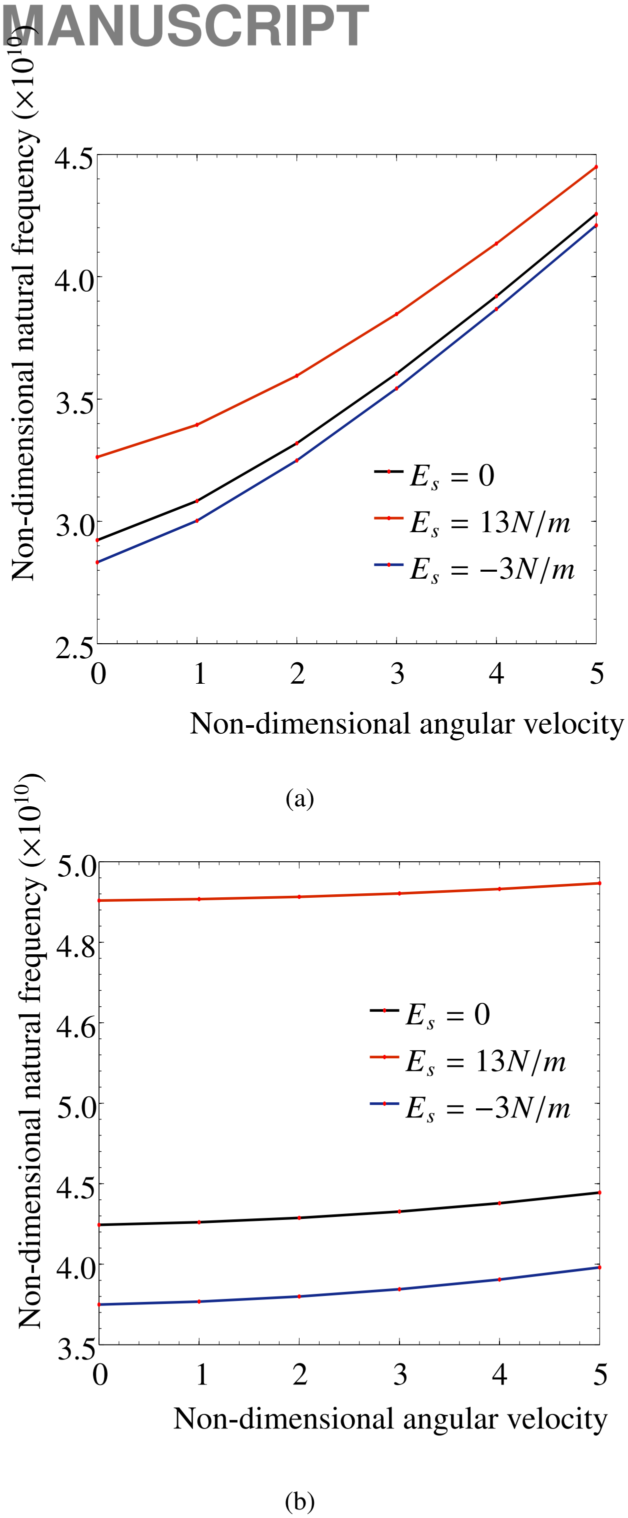

Figure 9: Plot of non-dimensional angular velocity versus non-dimensional natural frequency for different values of surface modulus E_s for (a) isotropic beam and (b) laminated beam.

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