Local Boxicity and Maximum Degree

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Abstract

The *local boxicity* of a graph G, denoted by $\text{lbox}(G)$, is the minimum positive integer l such that G can be obtained using the intersection of k (, where $k \geq l$,) interval graphs where each vertex of G appears as a non-universal vertex in at most l of these interval graphs. Let G be a graph on n vertices having m edges. Let Δ denote the maximum degree of a vertex in G. We show that,

- $\text{lbox}(G) \leq 2^{13 \log^* \Delta} \Delta$. There exist graphs of maximum degree Δ having a local boxicity of $\Omega(\frac{\Delta}{\log \Delta})$.
- $\text{lbox}(G) \in O(\frac{n}{\log n})$. There exist graphs on n vertices having a local boxicity of $\Omega(\frac{n}{\log n})$.
- $\text{lbox}(G) \leq (2^{13 \log^* \sqrt{m}} + 2)\sqrt{m}$. There exist graphs with m edges having a local boxicity of $\Omega(\frac{\sqrt{m}}{\log m})$.
- the local boxicity of G is at most its *product dimension*. This connection helps us in showing that the local boxicity of the *Kneser graph* $K(n, k)$ is at most $\frac{k}{2} \log \log n$.

The above results can be extended to the *local dimension* of a partially ordered set due to the known connection between local boxicity and local dimension. Finally, we show that the *cubicity* of a graph on n vertices of girth greater than $g + 1$ is $O(n^{\frac{1}{\lfloor g/2 \rfloor}} \log n)$.

Keywords: local boxicity, local dimension, boxicity, poset dimension, cubicity, product dimension, girth.

1. Introduction

A k-dimensional box or a k-*box* is defined as the Cartesian product of closed intervals $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$. A k-*box representation* of a graph G is a mapping of the vertices of G to k-boxes in the k-dimensional Euclidean space such that two vertices in G are adjacent if and only if their corresponding k-boxes have a non-empty intersection.

Definition 1 (Boxicity of a Graph)**.** *The boxicity of a graph G, denoted by* $\text{box}(G)$*, is the minimum positive integer k such that G has a k-box representation.*

A graph is an *interval graph* if it has a 1-box representation. An *interval representation* or 1-box representation of an interval graph I is a mapping f of each vertex in I to a closed interval on the real line in such a way that $\forall u, v \in V(I), uv \in E(I)$ if and only if $f(u)$ intersects $f(v)$. It is known that an interval graph may have multiple interval representations. Let $G = (V, E)$ be any graph and G_i, $1 \leq i \leq k$ be graphs on the same vertex set as G such that $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_k)$. Then G is the *intersection graph of G_is* and denoted by the equation $G = \cap_{i=1}^{k} G_i$. Projecting the k-boxes in a k-box representation of a graph into various axes yields an alternate definition of the boxicity of a graph which can be stated in terms of intersection of interval graphs.

Definition 2 (Alternate definition of boxicity, Roberts [35])**.** *The boxicity of a graph G is the minimum positive integer k such that G is the intersection graph of k interval graphs.*

It follows from the above definition of boxicity that if $G = \cap_{i=1}^{k} G_i$, for some graphs G_i, then $\text{box}(G) \leq \sum_{i=1}^{k} \text{box}(G_i)$. The notion of boxicity was first introduced by Roberts [35] in the year 1969. Cozzens [16] showed that computing the boxicity of a graph is NP-hard. Kratochvil [30] showed that determining whether the boxicity of a graph is at most two is NP-complete. See [12, 1, 11, 36] for more results on the boxicity of a graph.

1.1. Local boxicity of a graph

This paper revolves around a newly introduced graph parameter, *local boxicity*, which is a variant of boxicity. The notion of local boxicity was first introduced by Bläsius, Stumpf, and Ueckerdt [9]. An *l-dimensional local box* or an *l-local box* is defined as the cartesian product of intervals, $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_k$, $k \geq l$, where at least $k - l$ of these intervals are equal to the entire real line $\mathbb{R}$.

Definition 3 (l-local box representation)**.** *A l-local box representation of a graph G is a mapping of the vertices of G to l-local boxes in the k-dimensional Euclidean space, $l \leq k$, such that two vertices of G are adjacent in G if and only if their corresponding l-local boxes have a non-empty intersection.*

Definition 4 (Local boxicity of a graph)**.** *The local boxicity of a graph G, denoted by* $\text{lbox}(G)$*, is the minimum positive integer l such that G has an l-local box representation.*

Projecting the l-local boxes in an l-local box representation of a graph into various axes yields the following alternate definition of the local boxicity of a graph.

Definition 5 (Alternate definition of local boxicity)**.** *Local boxicity of a graph G, denoted by* $\mathrm{lbox}(G)$*, is the smallest positive integer l such that $G = \cap_{i=1}^{k} I_i$, where each I_i is an interval graph and each vertex of G appears as a non-universal vertex in at most l interval graphs in the collection $\{I_1, I_2, \ldots, I_k\}$.*

It follows from their definitions that $\mathrm{lbox}(G) \leq \mathrm{box}(G)$. The notion of local boxicity is useful in space efficient representation (or storage) of dense graphs having small local boxicity. As an example, consider Roberts' graph (Roberts' graph R_n is the graph obtained by removing a perfect matching from a complete graph on $2n$ vertices) having boxicity n. Representing this graph by storing every interval graph whose intersection is the original graph requires $\Omega(n^2)$ space. Any conventional representation of the Roberts' graph using an adjacency matrix or adjacency list also requires $\Omega(n^2)$ space. However, since the local boxicity of this graph is 1, there is a way to represent it in $O(n \log n)$ space. A graph G having local boxicity l can be represented or stored in the following manner. For each vertex v in the graph, an array $B_v[l][3]$ of size $l \times 3$ is maintained whose entries are as follows. For $1 \leq i \leq l$, $B_v[i][1] =$ identifier of the i^{th} interval graph that contains v as a non-universal vertex, $B_v[i][2] =$ left endpoint of the interval corresponding to v in the i^{th} interval graph where v is present as a non-universal vertex, and $B_v[i][3] =$ right endpoint of the interval corresponding to v in the i^{th} interval graph where v is present as a non-universal vertex. Each of the entries of this array requires $O(\log n)$ bits, where n is the number of vertices in G. Hence, the total space required to represent G is in $O(nl \log n)$. Note that if l is a constant (i.e. local boxicity of G is constant) then we get a space efficient representation in $O(n \log n)$ space.

1.2. Dimension and local dimension of a poset

Here we introduce the reader to the notions of the dimension and the local dimension of a partially ordered set and then in Section 1.3 show its relation with the graph parameters boxicity and local boxicity introduced above. A partially ordered set or *poset* $\mathcal{P} = (X, \preceq)$ is a tuple, where X denotes a finite or infinite set, and $\preceq$ is a binary relation on the elements of X. The binary relation $\preceq$ is reflexive, anti-symmetric and transitive. For any two elements $x, y \in X$, x is said to be *comparable* with y if either $x \preceq y$ or $y \preceq x$. If two elements $x, y \in X$ are comparable then either (x, y) is an ordered pair in $\mathcal{P}$ if $x \preceq y$ or (y, x) is an ordered pair in $\mathcal{P}$ if $y \preceq x$. Otherwise, if neither $a \preceq b$ nor $b \preceq a$, then a and b are called *incomparable elements* of $\mathcal{P}$. In this paper, we only deal with finite posets. A *linear order* is a partial order where every two elements are comparable with each other. A linear order is also called a *chain* in the literature. If a partial order $\mathcal{P} = (X, \preceq)$ and a linear order $L = (X, \preceq)$ are both defined on the same set X, and if every ordered pair in $\mathcal{P}$ is also present in L, then L is called a *linear extension* of $\mathcal{P}$. A collection of linear orders, say

$\mathcal{L} = \{L_1,\ L_2,\ \ldots,\ L_k\}$ with each L_i defined on X, is said to *realize* a poset $\mathcal{P} = (X, \preceq)$ if, for every two distinct elements $x, y \in X$, $x \preceq y \in \mathcal{P}$ if and only if $x \preceq_{L_i} y$, $\forall L_i \in \mathcal{L}$, where $x \preceq_{L_i} y$ denotes that $x \preceq y$ is in L_i. We call $\mathcal{L}$ a *realizer* for $\mathcal{P}$. The *dimension of a poset* $\mathcal{P}$, denoted by $\dim(\mathcal{P})$, is defined as the minimum cardinality of a realizer for $\mathcal{P}$. The concept of poset dimension was first introduced by Dushnik and Miller [17] and was extensively studied by researchers since then [38, 18, 25, 36].

The notion of *local dimension* was introduced recently by Ueckerdt [40] at the *Order and Geometry Workshop, 2016.* Local dimension of a poset is a variation of the standard poset dimension. The definition of local dimension originates from the concepts studied by Bläsius, Peter Stumpf and Torsten Ueckerdt [9], and Knauer and Ueckerdt[28]. A *partial linear extension or ple* of a poset $\mathcal{P}$ is defined as a linear extension of any subposet of $\mathcal{P}$.

Definition 6 (Local Realizer)**.** *A local realizer of a poset $\mathcal{P}$ is a family $\mathcal{L} = \{L_1, L_2, \ldots, L_l\}$ of ple's of $\mathcal{P}$ such that following conditions hold.*

1. *If $x \preceq y$ in $\mathcal{P}$ then there exists at least one ple $L_i \in \mathcal{L}$ such that $x \preceq_{L_i} y$.*
2. *If x and y are two incomparable elements of the poset $\mathcal{P}$, then there exist ple's $L_i, L_j \in \mathcal{L}$ such that $x \preceq_{L_i} y$ and $y \preceq_{L_j} x$.*

Given a local realizer $\mathcal{L}$ of $\mathcal{P}$ and an element $x \in \mathcal{P}$, the frequency of x in $\mathcal{L}$, denoted by $\mu_x(\mathcal{L})$, is defined as the number of ple's in $\mathcal{L}$ that contain x as an element. The maximum frequency of a local realizer is denoted by $\mu(\mathcal{L}) = \max_{x \in \mathcal{P}} \mu_x(\mathcal{L})$.

Definition 7 (Local Dimension)**.** *The local dimension of a poset $\mathcal{P}$, denoted by* $\operatorname{ldim}(\mathcal{P})$*, is defined as* $\min \mu(\mathcal{L})$ *where the minimum is taken over all the local realizers $\mathcal{L}$ of $\mathcal{P}$.*

For a poset $\mathcal{P}$, it follows from their definitions that $\operatorname{ldim}(\mathcal{P}) \leq \dim(\mathcal{P})$. See [39, 27, 6, 10] for more on the local dimension of a poset. The notion of local dimension can be useful in space efficient representation (or storage) of dense posets having small local dimension. For example, consider the dense crown poset S_n (S_n is a height 2 poset with n maximal elements $b_1, b_2, \ldots, b_n$, n minimal elements $a_1, a_2, \ldots, a_n$, and $a_i \preceq b_j$ for $i \neq j$. S_n is considered as a standard example in the literature of poset dimension.) having $\dim(S_n) = n$. Representing such a poset by either storing every relation in the partial order or by storing a realizer requires $\Omega(n^2)$ space. However, since $\operatorname{ldim}(S_n) = 3$, there is a way to represent S_n in $O(n \log n)$ space. A poset $\mathcal{P}$ having local dimension, $\operatorname{ldim}(\mathcal{P}) = p$ can be represented in following manner. For each element x in the ground set of $\mathcal{P}$ a 2-D array $A_x[p][2]$ of size $p \times 2$ is maintained whose entries are the following. For $1 \leq i \leq p$, $A_x[i][1] =$ identifier of the i^{th} ple that contains x, and $A_x[i][2] =$ position of x in i^{th} ple. Each of the entries of this array requires $O(\log n)$ bits to store the information, where n is the number of elements in $\mathcal{P}$. Therefore, the total space required to represent $\mathcal{P}$ is in $O(np \log n)$. Note that if p is a constant (i.e. local dimension of $\mathcal{P}$ is constant) then we get a space efficient representation in $O(n \log n)$ space.

1.3. Local boxicity and local dimension

A simple undirected graph $G_{\mathcal{P}}$ is the underlying comparability graph of a poset $\mathcal{P} = (X, \preceq)$ if X is the vertex set of $G_{\mathcal{P}}$ and two vertices are adjacent in $G_{\mathcal{P}}$ if and only if they are comparable in $\mathcal{P}$. Let $\mathcal{P}$ be a poset and $G_{\mathcal{P}}$ be the underlying comparability graph of $\mathcal{P}$. Adiga, Bhowmick, and Chandran [1] proved the following theorem.

Theorem 1 (Adiga, Bhowmick, and Chandran [1])**.** *Let $\chi(G_{\mathcal{P}})$ be the chromatic number of $G_{\mathcal{P}}$ and $\chi(G_{\mathcal{P}}) \neq 1$. Then,* $\frac{\mathrm{box}(G_{\mathcal{P}})}{\chi(G_{\mathcal{P}})-1} \leq \dim(\mathcal{P}) \leq 2\ \mathrm{box}(G_{\mathcal{P}})$.

A similar result connecting $\mathrm{lbox}(G_{\mathcal{P}})$ and $\mathrm{ldim}(\mathcal{P})$ was shown by Ragheb [34].

Lemma 2 (Ragheb [34])**.**

$$\frac{\mathrm{lbox}(G_{\mathcal{P}})}{\chi(G_{\mathcal{P}})} \leq \mathrm{ldim}(\mathcal{P}) \leq 2\ \mathrm{lbox}(G_{\mathcal{P}}) + 1,$$

when $\chi(G_{\mathcal{P}}) \neq 1$.

1.4. Our contribution

We know that the local boxicity of a graph is at most its boxicity. Thus all the known upper bounds for the boxicity of a graph also hold for its local boxicity. In this manuscript, we show some improved upper bounds for local boxicity. We prove the following results about the local boxicity of a graph.

- Finding an upper bound for the boxicity of a graph solely in terms of its maximum degree has been extensively studied [11, 20, 1]. Let $\mathrm{box}(\Delta)$ (or $\mathrm{lbox}(\Delta)$) denote the maximum boxicity (or local boxicity) of a graph having maximum degree Δ. Very recently, Scott and Wood [36] showed that $\mathrm{box}(\Delta) = O(\Delta \log^{1+o(1)} \Delta)$. It is known due to Erdős, Kierstead, and Trotter [18] and Adiga, Bhowmick and Chandran [1] that $\mathrm{box}(\Delta) = \Omega(\Delta \log \Delta)$. As for local boxicity, we show in Section 2.1 that $\mathrm{lbox}(\Delta) \leq 2^{13 \log^* \Delta}\Delta$. Further, with the help of the result due to Kim et al. [27] on local dimension, we show that $\mathrm{lbox}(\Delta) = \Omega(\frac{\Delta}{\log \Delta})$.[1]
- Bounding boxicity of *line graphs* in terms of its maximum degree has been extensively studied in the literature. The results of Alon et al. [4] and Scott and Wood [37] imply that the maximum local boxicity of a line graph of maximum degree Δ is $\Theta(\Delta)$. We know that line graphs belong to the class of *claw-free graphs*. Using an easy constructive proof, we show in Section *2.3* that the local boxicity of a claw-free graph is at most 2Δ. We have an algorithm that gives a 2Δ-local box representation in $O(n\Delta^2)$ time, where n is the number of vertices of the claw-free graph under consideration.

[1]Very recently, Esperet and Lichev [23] showed that $\mathrm{lbox}(\Delta) \in \Theta(\Delta)$.

- Esperet [21] showed that every graph on m edges has boxicity $O(\sqrt{m \log m})$. Further, in the same paper it was shown that this bound is asymptotically tight. In Section *2.2*, we show that the local boxicity of a graph is at most $(2^{13 \log^* \sqrt{m}}+2)\sqrt{m}$. We show the existence of graphs with m edges having a local boxicity of $\Omega(\frac{\sqrt{m}}{\log m})$.

- Roberts [35] showed the the maximum boxicity of a graph on n vertices is $\Theta(n)$. In Section 3, we show that the maximum local boxicity of a graph on n vertices is $\Theta(\frac{n}{\log n})$.

- In Section *4*, we show that the local boxicity of a graph is bounded from above by its *product dimension*. This connection helps us in showing that the local boxicity of the *Kneser graph* $K(n,k)$ is at most $\frac{k}{2} \log \log n$.

The table below summarizes our results on the local boxicity of a graph listed above. With the help of Lemma 2 that relates the parameters local dimension and local boxicity, every upper bound that we establish for local boxicity can be extended to local dimension.

Maximum boxicity of G	**Maximum local boxicity of G**	**Remarks**
$O(\Delta \log^{1+o(1)} \Delta)$, see [36]	$2^{13 \log^* \Delta}\Delta$, see Corollary 8	Δ is the max degree of G
$\Omega(\Delta \log \Delta)$, see [18]	$\Omega(\frac{\Delta}{\log \Delta})$, see Corollary 8	Δ is the max degree of G
$O(\sqrt{m \log m})$, see [21]	$(2^{13 \log^* \sqrt{m}} + 2)\sqrt{m}$, see Corollary 9	m denotes the no. of edges in G
$\Omega(\sqrt{m \log m})$, see [21]	$\Omega(\frac{\sqrt{m}}{\log m})$, see Corollary 9	m denotes the no. of edges in G
$\Theta(n)$, see [35]	$\Theta(\frac{n}{\log n})$, see Corollary 17	n denotes the no. of vertices in G
–	$\leq d$, see Theorem 18	d is the product dimension of G
–	$\leq \frac{k}{2} \log \log n$, see Corollary 21	when G is the Kneser graph $K(n,k)$

Finally, in Section *5*, we study *cubicity* (a parameter about cube representation of graphs. See Section 5 for the definition of cubicity.) of graphs of high girth. The boxicity of a graph is known to be at most its cubicity. We show that the cubicity of a graph on n vertices having girth greater than $g+1$ is $O(n^{\frac{1}{\lfloor g/2 \rfloor}} \log n)$.

1.5. Notations used

Unless mentioned explicitly, all logarithms used in the paper are to the base 2. For any positive integer n, we use $[n]$ to denote $\{1, \ldots, n\}$. Given a graph G, we shall use $V(G), E(G)$, and $\Delta(G)$ to denote its vertex set, edge set, and maximum degree, respectively. For any $v \in V(G)$, we use $N_G(v)$ to denote the neighborhood of v, i.e., $N_G(v) = \{u \in V(G) \mid vu \in E(G)\}$. Given a graph G, we use $G[S]$ to denote the subgraph induced on G by the vertex set S for any $S \subseteq V(G)$. Similarly, we use $G[S \cup T]$ to denote the subgraph induced on G by the vertex set $S \cup T$ for any $S, T \subseteq V(G)$. Let $G[S,T]$ denote the bipartite subgraph of the graph G with vertex set, $V(G[S,T]) = (S \cup T)$ and edge set, $E(G[S,T]) = E(G[S \cup T]) \setminus (E(G[S]) \cup E(G[T]))$.

2. Local boxicity, local dimension, and maximum degree

In this section, we discuss the connection between the local boxicity of a graph and its maximum degree. We begin by proving the following generalized lemma for computing the local boxicity of a graph whose vertex set is partitioned into disjoint parts.

Lemma 3. *Consider a graph G whose vertex set is partitioned into k parts, namely $V_1, V_2, \ldots, V_k$. Let $\max_{1 \le i < j \le k}(\text{lbox}(G[V_i \cup V_j])) = s$. Then,* $\text{lbox}(G) \le (k-1)s$.

Proof. Let $\mathcal{I}_{i \cup j}$ denote a collection of interval graphs that corresponds to a s-local box representation of $G[V_i \cup V_j]$ where the vertices $v \notin V_i, V_j$ are universal in all the interval graphs in the collection. That is, $G[V_i \cup V_j] = \bigcap_{I \in \mathcal{I}_{i \cup j}} I$. It is easy to see that G can be represented as follows.

$$G = \bigcap_{1 \le i < j \le k} \Big(\bigcap_{I \in \mathcal{I}_{i \cup j}} I \Big).$$

In such a representation, every vertex $v \in V_i$ appears as a universal vertex in every $I \in \mathcal{I}_{a \cup b}$, where $i \notin \{a, b\}$. Therefore, the maximum number of times a vertex $v \in V(G)$ is present as a non-universal vertex in the collection $\mathcal{I}_{i \cup j}$, where $1 \le i < j \le k$, is $(k-1)s$. Hence, the local boxicity of G, $\text{lbox}(G) \le (k-1)s$. □

2.1. Upper bound in terms of maximum degree

The following partitioning lemma is due to Alon et al. [37].

Lemma 4. *(Lemma 3 in [4]) For a graph G with maximum degree $\Delta \ge 2^{64}$, there exists a partition of $V(G)$ into r parts, where $r = \lceil \frac{400\Delta}{\log \Delta} \rceil$, such that for every vertex $v \in V(G)$ and for every part $V_i, i \in [r]$, $|N_G(v) \cap V_i| \le \frac{1}{2} \log \Delta$.*

We define $\text{lbox}(\Delta) := max\{\text{lbox}(G) : maximum\ degree\ of\ G\ is\ \Delta\}$.

Theorem 5. *For every positive integer Δ,* $\text{lbox}(\Delta) \le 2^{13 \log^* \Delta} \Delta$.

Proof. It is easy to see that the local boxicity of a graph with maximum degree 1 is at most 1. Assume $\Delta > 1$. For such graphs, we prove by induction on Δ that their boxicity is at most $2^{13 \log^* \Delta} \Delta$.
Base Case: $1 < \Delta \le 2^{64}$. Esperet [20] showed that for a graph G with maximum degree Δ, $box(G) \le \Delta^2 + 2$. When $1 < \Delta \le 2^{64}$, one can verify that $\Delta \le 2^{13 \log^* \Delta - 1}$. We thus have $\Delta^2 + 2 < 2\Delta^2 \le 2\Delta \cdot 2^{13 \log^* \Delta - 1} \le 2^{13 \log^* \Delta} \Delta$.
Induction step: Let $\Delta > 2^{64}$ and assume the theorem is true for every graph with maximum degree less than Δ. Let G be a graph with maximum degree Δ. We partition $V(G)$ into r parts, namely $V_1, V_2, \ldots, V_r$, where $r = \lceil \frac{400\Delta}{\log \Delta} \rceil$ and $|N_G(v) \cap V_i| \le \frac{1}{2} \log \Delta$, $\forall v \in V(G), i \in [r]$. Existence of such a partition is guaranteed by *Lemma 4*. For any $i, j \in [r]$, since the maximum degree of

$G[V_i \cup V_j]$ is at most $\log\Delta$, $\mathrm{lbox}(G[V_i \cup V_j]) \leq \mathrm{lbox}(\log\Delta)$. Applying Lemma 3, we get

$$\begin{aligned}
\mathrm{lbox}(\Delta) &\leq (r-1)\ \mathrm{lbox}(\log\Delta) \\
&< \left\lceil \frac{400\Delta}{\log\Delta} \right\rceil\ \mathrm{lbox}(\log\Delta) \\
&\leq 2^{13}\ \frac{\Delta}{\log\Delta}\ \mathrm{lbox}(\log\Delta) \\
&\leq 2^{13}\ \frac{\Delta}{\log\Delta}\ 2^{13\log^*(\log\Delta)}\ \log\Delta \\
&= 2^{13\log^*\Delta}\Delta.
\end{aligned}$$

□

Theorem 6 (Theorem 2, Kim et al. [27])**.** *The maximum local dimension of a poset on n points is* $\Theta(n/\log n)$.

Let $\mathrm{ldim}(\Delta) := \max\{\mathrm{ldim}(\mathcal{P}) : \text{maximum degree of } G_{\mathcal{P}} \text{ is } \Delta\}$ where $G_{\mathcal{P}}$ is the underlying comparability graph of a poset $\mathcal{P}$. As the maximum degree of $G_{\mathcal{P}}$. Corollary 7 follows directly from Lemma 2, Theorem 5, and Theorem 6.

Corollary 7. $\mathrm{ldim}(\Delta) \in \Omega(\frac{\Delta}{\log\Delta})$. *Further,* $\mathrm{ldim}(\Delta) \leq 2^{1+13\log^*\Delta}\Delta + 1$.

Corollary 8 follows directly from Lemma 2, Theorem 5, and Corollary 7.

Corollary 8. $\mathrm{lbox}(\Delta) \in \Omega(\frac{\Delta}{\log\Delta})$. *Further,* $\mathrm{lbox}(\Delta) \leq 2^{13\log^*\Delta}\Delta$.

Bridging the gap between the upper and the lower bound for $\mathrm{lbox}(\Delta)$ given in Corollary 8 is certainly an interesting question. Very recently, Esperet and Lichev [23] have shown that $\mathrm{lbox}(\Delta) \in \Theta(\Delta)$.

2.2. Local boxicity and the size of a graph

Esperet [21] showed that every graph on m edges has boxicity $O(\sqrt{m\log m})$. Further, in the same paper it is shown that this bound is asymptotically tight. In this section we explore how the local boxicity of a graph is connected with its size.

Corollary 9. *For a graph G having m edges,* $\mathrm{lbox}(G) \leq (2^{13\log^*\sqrt{m}} + 2)\sqrt{m}$. *Further, there exists a graph with m edges whose local boxicity is* $\Omega(\frac{\sqrt{m}}{\log m})$.

Proof. Let V' denote the set of vertices having degree at least $\sqrt{m}$ in G. We have, $|V'| \leq \frac{2m}{\sqrt{m}} = 2\sqrt{m}$. Each vertex in $G[V(G)\setminus V']$ has degree at most $\sqrt{m}$ in G. From Corollary 8 we have, $\mathrm{lbox}(G[V(G)\setminus V']) \leq 2^{13\log^*\sqrt{m}}\sqrt{m}$. Therefore, $\mathrm{lbox}(G) \leq \mathrm{lbox}(G[V(G)\setminus V']) + 2\sqrt{m} \leq 2^{13\log^*\sqrt{m}}\sqrt{m} + 2\sqrt{m} = (2^{13\log^*\sqrt{m}} + 2)\sqrt{m}$. Lemma 18 by Kim et al. [27] shows the existence of a height 2 poset $\mathcal{P}$, whose comparability graph has n vertices and $\Omega(n^2)$ edges, having $\mathrm{ldim}(\mathcal{P}) \in \Omega(\frac{n}{\log n})$. Thus, $\mathrm{ldim}(\mathcal{P}) \in \Omega(\frac{\sqrt{m}}{\log m})$ and, by Lemma 2, $\mathrm{lbox}(G_{\mathcal{P}}) \in \Omega(\frac{\sqrt{m}}{\log m})$. □

Corollary 10. *Let $\mathcal{P}$ be a poset whose underlying comparability graph has m edges. Then,* $\text{ldim}(\mathcal{P}) \leq (2^{13\log^* \sqrt{m}} + 2)2\sqrt{m} + 1$.

2.3. Constructing local box representations for claw-free graphs

Chandran, Mathew, and Sivadasan [14] showed that the boxicity of the line graph of a graph G with maximum degree Δ is $O(\Delta \log\log \Delta)$. The *line graph* $L(G)$ of a graph G is the graph with $V(L(G)) = E(G)$ and $E(L(G)) = \{ef : e, f \in E(G)$, e and f share a common endpoint$\}$. Later Alon et al. [4] improved this bound to $O(2^{9\log^* \Delta}\Delta)$, where $\log^* \Delta$ denotes the iterated logarithm of Δ, i.e., the number of times the log function is applied to get a result less than or equal to 1. Scott and Wood [37] showed that $\text{box}(L(G))$ of a graph G with maximum degree Δ is at most 20Δ, which is best possible upto a constant factor. Thus, boxicity of line graphs have been extensively studied. In this section, we study the local boxicity of claw-free graphs, a class of graphs which contains line graphs. A *claw graph* is a complete bipartite graph $K_{1,3}$ with one part containing a single vertex and the other part containing three vertices. A *claw-free graph* is a graph that contains no claw graph as its induced subgraph. We show that the local boxicity of a claw-free graph having a maximum degree of Δ is at most 2Δ. Our proof yields an algorithm for constructing 2Δ-local box representation for such graphs in $O(n\Delta^2)$ time.

Theorem 11. *Let G be a claw-free graph and let $\chi(G)$ denote its chromatic number. Then,* $\text{lbox}(G) \leq 2(\chi(G) - 1)$.

Proof. Based on an optimal vertex coloring of G, partition $V(G)$ into $\chi(G)$ color classes, namely $V_1, V_2, \ldots, V_{\chi(G)}$. From Lemma 3 we know $\text{lbox}(G) \leq (\chi(G) - 1) \max_{1 \leq i < j \leq \chi(G)} (\text{lbox}(G[V_i \cup V_j]))$. Since G is claw-free, $G[V_i \cup V_j]$ has a maximum degree of at most 2. That is, $G[V_i \cup V_j]$ is a disjoint union of paths and cycles. The local boxicity of a path is 1 and that of a cycle is 2. Thus, $\text{lbox}(G) \leq 2(\chi(G) - 1)$. □

For a graph G with maximum degree Δ, since $\chi(G) \leq \Delta + 1$, we have the following corollary.

Corollary 12. *Let G be a claw-free graph of maximum degree Δ. Then,* $\text{lbox}(G) \leq 2\Delta$.

In the proof of Theorem 11, a 2-local box representation for $G[V_i \cup V_j]$ can be obtained in $O(n)$ time, where n is the number of vertices in G. Thus, the proof of Theorem 11 yields an algorithm for constructing a 2Δ-local box representation for G that runs in $O(n\Delta^2)$ time.

3. Local boxicity and the order of a graph

It is known that the maximum boxicity of a graph on n vertices is $\Theta(n)$ (see [35]). In this section we show that the maximum local boxicity of a graph on n vertices is $\Theta(\frac{n}{\log n})$, where n is the order of a graph. The following theorem that partitions the edges of a graph into complete bipartite graphs is due to Erdős and Pyber. We use it in the proof of Theorem 14

Theorem 13 (Theorem 1, Erdős and Pyber [19])**.** *Let G be a graph on n vertices. The edge set $E(G)$ can be partitioned into complete bipartite graphs such that each vertex $v \in V(G)$ is contained in most $c \cdot \frac{n}{\log n}$ of the bipartite subgraphs.*

Theorem 14. *Let G be a graph on n vertices. Then,* $\text{lbox}(G) \leq c \cdot \frac{n}{\log n}$.

Proof. Let $\overline{G}$ be the complement of the graph G i.e. the vertex set $V(\overline{G}) = V(G)$ and the edge set $E(\overline{G}) = \{uv : uv \notin E(G)\}$. Now, $E(\overline{G})$ is partitioned into k complete bipartite graphs $G_1, G_2, \ldots, G_k$ using Theorem 13. From each complete bipartite graph G_i we construct one interval graph I_i whose interval representation is denoted by f_i. Let A_i and B_i be the two constituting parts of the complete bipartite graph G_i. All the vertices $v \in A_i$ are assigned the interval $[1, 2]$ and all the vertices $v \in B_i$ are assigned the interval $[3, 4]$ in f_i. Any vertex that is not present in G_i is assigned the entire real line as its interval in f_i.

Note that each vertex of G appears as a non-universal vertex in at most $c \cdot \frac{n}{\log n}$ number of interval graphs from the set $\{I_1, I_2, \ldots, I_k\}$. We now argue that $G = \cap_{i=1}^{k} I_i$.

Claim 15. *If $uv \notin E(G)$ then there exists exactly one interval graph I_i where $uv \notin E(I_i)$.*

As u and v are not adjacent in G, they are adjacent in $\overline{G}$. Since we have partitioned $E(\overline{G})$ into k complete bipartite graphs, there exists exactly one complete bipartite graph, say G_i, which has, without loss of generality, $u \in A_i, v \in B_i$. Then, u receives the interval $[1, 2]$ and v receives the interval $[3, 4]$ in f_i. Thus, $uv \notin E(I_i)$. This proves the claim.

Claim 16. *If $uv \in E(G)$ then in all the interval graphs I_i, where $1 \leq i \leq k$, $uv \in E(I_i)$.*

Since $uv \in E(G)$, they are not adjacent in $\overline{G}$. In every complete bipartite graph G_i that u or v is absent, it (u or v) acts as a universal vertex in the interval graph I_i constructed. Further, in the bipartite graphs G_i where both u and v are present, they appear on the same part (A_i or B_i) thus getting the same interval ($[1, 2]$ or $[3, 4]$) in f_i. Hence, $uv \in E(I_i), \forall 1 \leq i \leq k$. This proves the claim and thereby the theorem.

□

Combining Lemma 2, Theorem 6 and Theorem 14, we get the following corollary.

Corollary 17. *The maximum local boxicity of a graph on n vertices is* $\Theta(\frac{n}{\log n})$.

4. Local boxicity and the product dimension of a graph

The *direct product*, denoted by $G \times H$, of graphs G and H has $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(a,b)(c,d) : a, c \in V(G),\ b, d \in V(H),\ ac \in E(G), \text{ and } bd \in E(H)\}$. The *product dimension*, also known as *Prague dimension*, of a graph G is the minimum positive integer k such that G is an induced subgraph of the direct product of k complete graphs. The parameter product dimension was introduced and studied by Nešetřil and Pultr [32]. More results in [24, 31]. Researchers have tried to find relations between various dimensional parameters of a graph. Theorem 1 relates the dimension of a poset and the boxicity of its comparability graph; Füredi [24] tries to explore a connection with the poset dimension to bound the product dimension of a Kneser graph; Chatterjee and Ghosh [15] finds a relation between the *Ferrers dimension* of a graph and its boxicity; Basavaraju et al. [7] relates the *separation dimension* of a graph with the boxicity of its line graph. Though a relation between the boxicity of a graph and its product dimension is not known to be explored yet, it may be noted that Chandran et al. [13] studied boxicity of the (Cartesian, strong, and direct) products of graphs. Theorem 14 in their paper gives a trivial upper bound of nk for the boxicity of a graph with n vertices having a product dimension of k. In this section, we show that the product dimension of a graph is an upper bound to its local boxicity. For this, we state below an alternate definition of product dimension by Lovász, Nešetřil, and Pultr [31].

Definition 8 (Lovász, Nešetřil, and Pultr [31])**.** *The product dimension of a graph G, denoted by* $\mathrm{pdim}(G)$*, is the minimum positive integer k for which there exists a function $f : V(G) \to \mathbb{N}^k$, such that $uv \in E(G)$, if and only if $f(u)$ and $f(v)$ differ in exactly k coordinates.*

Theorem 18. *For any graph G,* $\mathrm{lbox}(G) \leq \mathrm{pdim}(G)$.

Proof. Let $\mathrm{pdim}(G) = k$. Let $f : V(G) \to \mathbb{N}^k$ be a k-coordinate representation of G (that satisfies the condition of Definition 8) where $f_i(v)$ denotes the i^{th} coordinate of $f(v)$. For each $i \in [k]$, let $S_i = \{f_i(v) : v \in V(G)\}$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. For each $i \in [k], j \in S_i$, we construct an interval graph $I_{i,j}$ in the following way. Let $g_{i,j}$ be an interval representation of $I_{i,j}$ ($g_{i,j} : V(I_{i,j}) \to X$, where X is the set of all closed intervals on the real line). For a vertex $v_a \in \{v_1, v_2, \ldots, v_n\}$, if $f_i(v_a) = j$, then $g_{i,j}(v_a) = [a, a]$. Otherwise, $g_{i,j}(v_a) = [1, n]$.

In order to show that $G = \bigcap_{i\in[k]}(\bigcap_{j\in S_i} I_{i,j})$, consider any $v_a, v_b \in V(G)$. If $v_a v_b \in E(G)$, then $\forall i \in [k], f_i(v_a) \neq f_i(v_b)$ and therefore either v_a or v_b is a universal vertex in every interval graph we construct. Now, suppose $v_a v_b \notin E(G)$. Then, for some $i \in [k]$, $f_i(v_a) = f_i(v_b)$ $(= j$, say) and therefore, from our construction, the intervals of v_a and v_b don't overlap in $g_{i,j}$. Thus, $G = \bigcap_{i \in [k]} (\bigcap_{j \in S_i} I_{i,j})$.

For any given $i \in [k]$, a vertex $v \in V(G)$ appears as a non-universal vertex in exactly one interval graph $I_{i,j}$ where $j = f_i(v)$. Thus, v appears as a non-universal vertex in at most k interval graphs in this collection. Hence, $\mathrm{lbox}(G) \leq k = \mathrm{pdim}(G)$. □

Corollary 19. $\mathrm{ldim}(\mathcal{P}) \leq 2\ \mathrm{pdim}(G_{\mathcal{P}}) + 1$, *where* $\mathcal{P} = (X, \preceq)$ *is a poset and* $G_{\mathcal{P}}$ *is its underlying comparability graph.*

The *Kneser graph* $K(n, k)$ is the graph whose vertex set is the set of all k-sized subsets of $[n]$ and any two such vertices are adjacent to each other if and only if the corresponding k-sized subsets do not intersect with each other. The following result is due to Gargano, Körner, and Vaccaro [26] and Poljak, Pultr, and Rödl [33]. It is stated in Section VIII of Korner and Orlitsky [29].

Theorem 20 (Gargano, Körner, and Vaccaro [26], Poljak, Pultr, and Rödl [33])**.**

$$\log\log n \leq \mathrm{pdim}(K(n, k)) \leq \frac{k}{2} \log\log n.$$

Corollary 21 follows directly from Theorem 18 and Theorem 20.

Corollary 21.

$$\mathrm{lbox}(K(n, k)) \leq \frac{k}{2} \log\log n.$$

It would be interesting to explore how tight this upper bound for the local boxicity of a Kneser graph is. It is known from [31] that the graph nK_2, that is the graph of $2n$ vertices formed by taking n copies of an edge, has a product dimension of $\Omega(\log n)$. However, the local boxicity of this graph is one. That means that one can not give a non-trivial lower bound for the local boxicity of a graph solely in terms of its product dimension. It is worth exploring whether one can show such a lower bound in terms of the product dimension and a third parameter of the graph under consideration.

5. Cube representation of graphs of high girth

The *girth* of a graph is the length of a smallest cycle in it. Girth of an acyclic graph is assumed to be ∞. A k-dimensional cube or a k-cube is defined as the Cartesian product of unit length closed intervals $[a_1, a_1 + 1] \times [a_2, a_2 + 1] \times \cdots \times [a_k, a_k + 1]$. Therefore, k-cubes are k dimensional axis-parallel cubes. A k-cube representation of a graph G is a mapping of the vertices of G to k-cubes in the k-dimensional Euclidean space such that two vertices in G are adjacent if and only if their corresponding k-cubes have a non-empty intersection.

Definition 9 (Cubicity of a graph)**.** *The cubicity of a graph* G*, denoted by* $\mathrm{cub}(G)$*, is defined as the minimum positive integer* k *such that* G *has a* k*-cube representation.*

A graph is a *unit interval graph* if it is an interval graph and it has an interval representation where every interval is of unit length. Below we state an alternate definition of cubicity in terms of unit interval graphs.

Definition 10 (Alternate definition of cubicity). *The cubicity of a graph G, denoted by $\mathrm{cub}(G)$, is the minimum positive integer k such that there exist k unit interval graphs $I_1, I_2, \ldots, I_k$ with $G = \bigcap_{i=1}^{k} I_i$.*

It is known and follows from their definitions that, for a graph G, $\mathrm{lbox}(G) \leq \mathrm{box}(G) \leq \mathrm{cub}(G)$. Most graphs of high boxicity (and, thereby high cubicity) we know are graphs of low girth, whether it be the Roberts' graph R_n defined in Section 1.1 (a complete graph on $2n$ vertices minus a perfect matching) or the random graph studied by Erdős, Kierstead, and Trotter [18]. Therefore, it is a natural question to ask whether the boxicity of a graph decreases as its girth increases. It was shown by Bhowmick and Chandran [8] that if G is an *asteroidal triple free* graph having girth at least 5, then the $\mathrm{box}(G) \leq 2$ and $\mathrm{cub}(G) \leq 2\lceil \log_2 \psi(G) \rceil + 4$, where $\psi(G)$ denotes the claw number of G. The *claw number* of a graph G is the number of edges in the largest star that is an induced subgraph of G. Esperet and Joret [22] showed that for a fixed surface Σ there exists an integer g_Σ such that every graph with girth at least g_Σ embeddable in Σ has boxicity at most 4. Here we give a general upper bound for the cubicity of a graph in terms of its girth and order. We show that, for a graph G on n vertices with girth greater than $g+1$, $\mathrm{cub}(G) \in O(n^{\frac{1}{\lfloor \frac{g}{2} \rfloor}} \log n)$. We first show in Lemma 23 that such a graph G is $\left\lceil n^{\frac{1}{\lfloor \frac{g}{2} \rfloor}} \right\rceil$-degenerate and then use Theorem 22 (due to Adiga, Chandran, and Mathew [3]) stated below, to obtain the result.

Definition 11. *A graph is k-degenerate if the vertices of the graph can be enumerated in such a way that every vertex is followed by at most k of its neighbors. The least number k such that the graph is k-degenerate is called the degeneracy of the graph.*

Theorem 22 (Theorem 1, Adiga, Chandran, and Mathew [3]). *For every k-degenerate graph G, $\mathrm{cub}(G) \leq (k+2)\lceil 2e \log n \rceil$.*

Lemma 23. *Let G be a graph on n vertices having girth greater than $g+1$. Then G is k-degenerate, where $k = \left\lceil n^{\frac{1}{\lfloor \frac{g}{2} \rfloor}} \right\rceil$.*

Proof. We will prove this lemma by contradiction. Suppose the lemma is not true. Then, there exists a $V_1 \subseteq V(G)$ such that $G[V_1]$ is connected and every vertex in $G[V_1]$ has degree greater than k. Consider a vertex v in $G[V_1]$. Let S be the set of vertices that are at a distance of at most $\lfloor \frac{g}{2} \rfloor$ from v. Since girth of $G[V_1]$ is greater than $g+1$, $G[S]$ is a tree. As the minimum degree of a vertex in $G[V_1]$ is greater than k, the leaf vertices of $G[S]$ have to be adjacent to some vertices in $V_1 \setminus S$. Thus, the set $V_1 \setminus S$ is non-empty. But then we have $|S| > k^{\lfloor \frac{g}{2} \rfloor} = n$, contradicting the fact that $V_1 \setminus S$ is non-empty. Hence our assumption that G is not k-degenerate is false. $\square$

From Lemma 23 and Theorem 22, we have the following theorem.

Theorem 24. *Let G be a graph on n vertices with girth greater than $g+1$. Then,* $\mathrm{cub}(G)$ *is in* $O(n^{\frac{1}{\lfloor \frac{g}{2} \rfloor}} \log n)$. *More precisely,* $\mathrm{cub}(G) \leq (n^{\frac{1}{\lfloor \frac{g}{2} \rfloor}} + 2)\lceil 2e \log n \rceil$.

Example 1. *It was shown by Adiga and Chandran [2] that the cubicity of a graph G is at least $\lceil \log_2 \psi(G) \rceil$, where $\psi(G)$ is the claw number of G. Consider the star graph $S_{1,n-1}$ on n vertices having a claw number of $n-1$. Since the girth of a tree is assumed to be ∞, Theorem 24 gives an asymptotically tight upper bound for the cubicity of $S_{1,n-1}$.*

Example 2. *Alon, Ganguly, and Srivastava [5] showed that there exists a graph G on n vertices having girth at least $\frac{log_5 n}{4}$. Consider such a graph G. Take a vertex v in G. We add n pendent vertices to v to construct a graph $G^{'}$. Thus, the claw number of $G^{'}$, $\psi(G^{'}) \geq n$. Hence,* $\mathrm{cub}(G^{'}) \in \Omega(\log n)$. *Theorem 24 gives an upper bound for the cubicity of $G^{'}$.*

Example 3. *Consider the bipartite graph G obtained by removing a perfect matching from a complete bipartite graph, $K_{n,n}$ on $2n$ vertices. It is known that the boxicity (and thereby cubicity) of G is at least $\frac{n}{2}$. Applying Theorem 24 with $g = 2$, we get* $\mathrm{cub}(G) \in O(n \log n)$.

From Example 1 and Example 2, we observe that there are graphs of high girth on n vertices whose cubicity is in $\Omega(\log n)$. From Example 3, we observe that for a graph on n vertices with girth greater than $g + 1$, we cannot get a bound of $O(n^{\alpha g})$ for its cubicity, where α is a constant less than $\frac{1}{2}$. From these two observations, we believe it would be worthwhile to try improving the bound in Theorem 24 to $(c_1 n^{\frac{1}{\lfloor \frac{g}{2} \rfloor}} + c_2 \log n)$, where c_1 and c_2 are constants.

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