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Level Sets of $(p, e-p)$ Outer Generalized Pseudo Spectrum

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Abstract Let $\mathscr{A}$ be a complex Banach algebra with unit e. Let p be a non trivial idempotent element in $\mathscr{A}$ and $\varepsilon > 0$. For $a \in \mathscr{A}$, it is proved that the interior of the level set of $(p, e-p)-\varepsilon$ pseudo spectrum of a is empty in the unbounded component of $(p, e-p)$ resolvent set of a. An example is constructed to show that the condition 'unbounded component' can not be dropped. Further, it is proved this 'unbounded component' can be dropped in the case when $\mathscr{A}$ is $B(X)$ where X is a complex uniformly convex Banach space. That is, if $T \in B(X)$ then interior of the level set of $(p, I-p)-\varepsilon$ pseudo spectrum is empty in $(p, I-p)$ resolvent set of T.

Keywords Analytic vector valued map, $(p,q)-\varepsilon$ pseudo spectrum, complex uniformly convex Banach space.

1 Introduction

Let f be a complex valued analytic function defined on an open connected subset Ω of $\mathbb{C}$. If f is non constant then by maximum modulus theorem, $|f|$ can not be constant on Ω. This need not to be true for general analytic Banach algebra valued functions. We first see the definition of analytic Banach algebra valued function. Let $\mathscr{A}$ be a complex Banach algebra with unit e and Ω be an open subset of $\mathbb{C}$.

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Definition 1 ([11], Definition 3.2) A map $f : \Omega \to \mathscr{A}$ is said to be differentiable at the point $\mu \in \Omega$ if there exists an element $f'(\mu) \in \mathscr{A}$ such that

$$\lim_{\lambda \to \mu} \left\| \frac{f(\lambda) - f(\mu)}{\lambda - \mu} - f'(\mu) \right\| = 0.$$

If f is differentiable at every point in Ω then f is said to be analytic in Ω.

Consider $\Omega = \mathbb{C}$, $\mathscr{A} = \mathbb{M}_2(\mathbb{C}) := \left\{ A : A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ where } a_{ij} \in \mathbb{C} \right\}$ with norm $\|A\|_1 = \max_{1 \le j \le 2} \left\{ \sum_{i=1}^{2} |a_{ij}| \right\}$. Define $\psi : \mathbb{C} \to \mathbb{M}_2(\mathbb{C})$ by $\psi(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. Clearly ψ is analytic and for any $\mu \in \mathbb{C}$, $\psi'(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Moreover $\|\psi(\lambda)\|_1$ is constant on the open set $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Hence, the norm of general Banach algebra valued analytic maps may be constant or need not to be constant in an open connected subset of $\mathbb{C}$. We shall identify $\lambda \cdot e$ as λ for any $\lambda \in \mathbb{C}$. Recall that for $a \in \mathscr{A}$, the resolvent set is defined as $\{\lambda \in \mathbb{C} : (a - \lambda) \text{ is invertible in } \mathscr{A}\}$ and it is denoted by $\rho(a)$. Complement of $\rho(a)$ is called spectrum of a, which is denoted by $\sigma(a)$. It is a well known fact that $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$ and hence $\rho(a)$ is a nonempty open subset of $\mathbb{C}$. J. Globevnik in [8] studied about the norm constant value of the map,

$$R : \rho(a) \to \mathscr{A} \text{ by } R(\lambda) = (a - \lambda)^{-1}$$

in the open subset of $\rho(a)$. He proved in [8] (Proposition 1, Proposition 2),

(a) for any $a \in \mathscr{A}$, $\left\|(a - \lambda)^{-1}\right\|$ can not attain local maximum in any unbounded component of $\rho(a)$.
(b) If X be a complex uniformly convex Banach space and $T \in B(X)$ then $\left\|(T - \lambda)^{-1}\right\|$ can not attain local maximum in any open subset of $\rho(T)$.

One can find, some more answers related to this question in [3], [4] and [5]. E. Shargorodsky in [13] (Theorem 3.1) showed, there exists an invertible bounded linear operator T acting on the Banach space $l^\infty(\mathbb{Z})$ with norm $\|x\|_* = \sup_{k \neq 0} |x_k| + |x_0|$ where $x = (x_k)_{k \in \mathbb{Z}}$, such that $\left\|(T - \lambda)^{-1}\right\|$ is constant in an open neighborhood of $\lambda = 0$.

The main aim of this paper is to investigate and classify the possible cases, when the norm of the (p,q) resolvent map (see Definition 3) is not constant in an open connected subset of the (p,q) resolvent set (see Definition 3).

Consider two idempotent elements $p, q \in \mathscr{A}$ i.e. $p^2 = p$ and $q^2 = q$.

Definition 2 ([10], Definition 1.1) Let $a \in \mathscr{A}$. An element $b \in \mathscr{A}$ satisfying,

$$bab = b, ba = p \text{ and } 1 - ab = q$$

will be called a (p,q) outer generalized inverse of a and it is denoted by $a^{(2)}_{p,q}$.

In [10], Kolundžija introduced the concept of the $(p,q)-\varepsilon$-pseudo spectrum of a in $\mathscr{A}$. Let $\varepsilon > 0$ and $a \in \mathscr{A}$. The $(p,q)-\varepsilon$-pseudo spectrum is defined as

$$\Lambda^{(2)}_{(p,q)-\varepsilon}(a) := \left\{\lambda \in \mathbb{C} : (a-\lambda)^{(2)}_{p,q} \text{ does not exist or } \left\|(a-\lambda)^{(2)}_{p,q}\right\| \geq \varepsilon\right\}.$$

In the same article, Kolundžija discusses about $(p,q)-\varepsilon$-pseudo spectrum of elements of the Banach algebra which are in the block matrix form. For the geometric understanding of $(p,q)-\varepsilon$-pseudo spectrum, because of the inequalities in $(p,q)-\varepsilon$-pseudo spectrum and in order to understand it, one has to know more about its boundary set. It is clear that the boundary sets are subsets of the set (see theorem 5),

$$L^{(2)}_{(p,q)-\varepsilon}(a) = \left\{\lambda \in \mathbb{C} : \left\|(a-\lambda)^{(2)}_{p,q}\right\| = \varepsilon\right\}.$$

The above set is called level set of $(p,q)-\varepsilon$ pseudo spectrum. In computational point of view, if we are sure that the level sets do not contain any interior point then it can help us to trace out the boundary sets of $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$. Because of the reasons so far discussed, this paper studies the interior property of $L^{(2)}_{(p,e-p)-\varepsilon}(a)$ for given $a \in \mathscr{A}$.

Preliminary section of this note concentrates on the non emptiness of $\Lambda^{(2)}_{(p,e-p)-\varepsilon}(a)$ and the analyticity of $(p,e-p)$ resolvent map. Section 3 of this paper focus on the interior property of the level set of $(p,e-p)-\varepsilon$ pseudo spectrum set. Theorems which are in this section (Theorem 6, Theorem 7) are extended version of the results of Globevnik. Using these results we prove $(p,q)-\varepsilon$ pseudo spectrum has finite number of components and each component has nonempty intersection with (p,q) spectrum (Theorem 8). Example is constructed to show that $L^{(2)}_{(p,e-p)-\varepsilon}(a)$ may have nonempty interior (Example 4) for some Banach algebra $\mathscr{A}$ and $a \in \mathscr{A}$.

Throughout this paper, $B(\lambda,r)$ denotes the open disk in the complex plane with center λ and radius $r > 0$ and $B(X)$ denotes the set of all bounded linear operators defined on the complex Banach space X.

2 Preliminaries

In this section, we introduce some basic definitions, terminologies and results which are related to $(p,e-p)$ resolvent set and $(p,e-p)-\varepsilon$ pseudo spectrum and the major goal is to show the non-emptiness of these sets.

Definition 3 For an element $a \in \mathscr{A}$, the (p,q)-resolvent set is defined as

$$\rho^{(2)}_{p,q}(a) := \left\{\lambda \in \mathbb{C} : (a-\lambda)^{(2)}_{p,q} \text{ exists}\right\}.$$

The complement of the set $\rho^{(2)}_{p,q}(a)$ is called (p,q)-spectrum and it is denoted by $\sigma^{(2)}_{p,q}(a)$. The map $\lambda \mapsto (a-\lambda)^{(2)}_{p,q}$ defined from $\rho^{(2)}_{p,q}(a)$ to $\mathscr{A}$ is called the (p,q)-resolvent map.

From now onwards, we consider the idempotent $p \neq 0$ and $p \neq e$ and we fix the idempotent element $q := e - p$. If $\lambda \in \rho^{(2)}_{p,q}(a)$ then we denote the element $(a-\lambda)^{(2)}_{p,q}$ by $R_a(\lambda)$.

Note 1 For given $a \in \mathscr{A}$, if $R_a(\lambda)$ exists for some $\lambda \in \mathbb{C}$ then from definition 2,

$$[R_a(\lambda)]\,(a-\lambda) = p \text{ and } (a-\lambda)\,[R_a(\lambda)] = p. \tag{1}$$

By equation (1), $ap = pa$. Consequently, if $ap \neq pa$ then $\sigma^{(2)}_{p,q}(a) = \mathbb{C}$. Because of this reason, in the rest of the paper we assume $ap = pa$ for given $a \in \mathscr{A}$.

Note 2 If $R_a(\lambda)$ exists for some $\lambda \in \mathbb{C}$ then by equation (1), $R_a(\lambda)$ and a commutes and $[(a-\lambda)^n]^{(2)}_{p,q}$ exists for any $n \in \mathbb{N}$. Moreover, $[(a-\lambda)^n]^{(2)}_{p,q} = [R_a(\lambda)]^n$.

Note 3 If $\lambda \in \sigma^{(2)}_{p,q}(a)$ then we assume that $\|R_a(\lambda)\| = \infty$.

It is well known that $\rho(a)$ for any $a \in \mathscr{A}$ is nonempty open subset of $\mathscr{A}$, the following lemma and Theorem prove the same for (p,q) resolvent set.

Lemma 1 *Let $a \in \mathscr{A}$. If $\lambda \in \rho(a)$ then $\lambda \in \rho^{(2)}_{p,q}(a)$*

Proof *It is easy to see that $R_a(\lambda) = p(a-\lambda)^{-1}$ for any $\lambda \in \rho(a)$*

Theorem 1 *The set $\rho^{(2)}_{p,q}(a)$ is a nonempty open subset of $\mathbb{C}$, for any $a \in \mathscr{A}$.*

Proof *By lemma 1, $\rho^{(2)}_{p,q}(a)$ is nonempty. Take $\mu \in \rho^{(2)}_{p,q}(a)$, for any $\lambda \in \mathbb{C}$ satisfies*

$$|\mu - \lambda| < \frac{1}{\|R_a(\mu)\|}$$

we have $e + [R_a(\mu)]\,((a-\lambda) - (a-\mu))$ is invertible. From equation (1),

$$(a-\lambda)\,[R_a(\mu)]\,(a-\mu) = (a-\mu)\,[R_a(\mu)]\,(a-\lambda).$$

Hence by Theorem 4.1 in [6], $\lambda \in \rho^{(2)}_{p,q}(a)$.

The following corollary shows the norm of the (p,q) resolvent is very large in the neighborhood of an element from the (p,q) spectrum set.

Corollary 1 *Let $\{\lambda_n\}$ be a sequence from $\rho^{(2)}_{p,q}(a)$. If $\lambda_n \to \lambda$ for some $\lambda \in \sigma^{(2)}_{p,q}(a)$ then $\|R_a(\lambda_n)\| \to \infty$.*

Proof *Suppose $\|R_a(\lambda_n)\| \leq M$ for some $M \in \mathbb{R}$ then $\dfrac{1}{\|R_a(\lambda_n)\|} \geq \dfrac{1}{M}$. Since $\lambda_n \to \lambda$, for the real number $\dfrac{1}{M+1}$, there exists $n_0 \in \mathbb{N}$ such that*

$$|\lambda - \lambda_n| < \frac{1}{M+1} < \frac{1}{M} \leq \frac{1}{\|R_a(\lambda_n)\|} \text{ for all } n \geq n_0.$$

By Theorem 1, $\lambda \in \rho^{(2)}_{p,q}(a)$. This is a contradiction.

Theorem 2 *The map $f:\rho^{(2)}_{p,q}(a)\to\mathscr{A}$ defined by $f(\lambda)=[R_a(\lambda)]^n$ is analytic for each $n\in\mathbb{N}$.*

Proof *We first prove this theorem for $n=1$. For any $\lambda,\mu\in\rho^{(2)}_{p,q}(a)$, by Theorem 4.2 (a) in [6],*

$$[R_a(\lambda)]-[R_a(\mu)]=(\lambda-\mu)[R_a(\lambda)][R_a(\mu)]. \tag{2}$$

Fix $\mu\in\rho^{(2)}_{p,q}(a)$ and consider the open set $B\left(\mu,\frac{1}{\|R_a(\mu)\|}\right)$. By Theorem 1, $B\left(\mu,\frac{1}{\|R_a(\mu)\|}\right)$ is a subset of $\rho^{(2)}_{p,q}(a)$. Since $e-[R_a(\mu)](\lambda-\mu)$ is invertible for any $\lambda\in B\left(\mu,\frac{1}{\|R_a(\mu)\|}\right)$ and from equation (2),

$$R_a(\lambda)=\sum_{n=0}^{\infty}(\lambda-\mu)^n[R_a(\mu)]^{n+1}.$$

Hence the map $\lambda\mapsto R_a(\lambda)$ is analytic. The map $\lambda\mapsto[R_a(\lambda)]^n$ is also analytic because it is the product of n analytic functions of the form $\lambda\mapsto[R_a(\lambda)]$.

The following are some examples of (p,q)-resolvent set and (p,q)-spectrum for given $a\in\mathscr{A}$ and $p\in\mathscr{A}$.

Example 1 Let $a=\lambda$ for some $\lambda\in\mathbb{C}$. It is easy to see, $\rho^{(2)}_{p,q}(a)=\mathbb{C}\setminus\{\lambda\}$ and $\sigma^{(2)}_{p,q}(a)=\{\lambda\}$.

Our next example shows that $\rho^{(2)}_{p,q}(a)$ may have multiple components.

Example 2 Consider the set $E=\{z\in\mathbb{C}:1\le|z|\le2\}\cup\{z\in\mathbb{C}:3\le|z|\le4\}$. Take the operator $T\in B\left(\ell^2(\mathbb{N})\right)$ with,

$$T(e_{2i-1})=r_ie_{2i-1} \text{ and } T(e_{2i})=q_ie_{2i} \text{ for all } i\in\mathbb{N}$$

where $\{e_i:i\in\mathbb{N}\}$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$, $\{r_i\in\mathbb{C}:i\in\mathbb{N}\}$ is countable dense subsets of $\{z\in\mathbb{C}:1\le|z|\le2\}$ and $\{q_i\in\mathbb{C}:i\in\mathbb{N}\}$ is a countable dense subset of $\{z\in\mathbb{C}:3\le|z|\le4\}$. Take the projection operator $P\in B\left(\ell^2(\mathbb{N})\right)$

$$P(e_{2i-1})=e_{2i-1} \text{ and } P(e_{2i})=0 \text{ for all } i\in\mathbb{N}$$

Take $Q=I-P$. It is evident that $PT=TP$ and $\sigma(T)=E$. By lemma 1,

$$\{z\in\mathbb{C}:|z|>4\}\cup\{z\in\mathbb{C}:2<|z|<3\}\cup\{z\in\mathbb{C}:|z|<1\}\subset\rho^{(2)}_{(P,Q)}(T).$$

We prove, $\{z\in\mathbb{C}:1\le|z|\le2\}\subseteq\sigma^{(2)}_{(P,Q)}(T)$. Suppose $R_T(r_i)$ exists for some r_i, then from the equation $[R_T(r_i)](T-r_i)=P$,

$$Ker(T-r_i)\cap Ran(P)=\{0\}.$$

where $Ker(T-r_i)$ denotes the null space of $T-r_i$ and $Ran(P)$ denotes the range space of P. But for every $i\in\mathbb{N}$,

$$e_{2i-1}\in Ker(T-r_i)\cap Ran(P)$$

which is a contradiction. Hence $\{r_i \in \mathbb{C} : i \in \mathbb{N}\} \subset \sigma^{(2)}_{(P,Q)}(T)$. Since $\{r_i \in \mathbb{C} : i \in \mathbb{N}\}$ is dense in $\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$ and $\rho^{(2)}_{(P,Q)}(T)$ is open,

$$\{z \in \mathbb{C} : 1 \leq |z| \leq 2\} \subseteq \sigma^{(2)}_{(P,Q)}(T)$$

From this we also conclude, $\rho^{(2)}_{(P,Q)}(T)$ has more than one component.

Our next objective is to prove, $(p,q)-\varepsilon$ pseudo spectrum is non empty. We achieve this with the aid of the results we observed so far.

Definition 4 ([10], Definition 3.3) Let $\varepsilon > 0$. The $(p,q)-\varepsilon$-pseudospectrum of an element $a \in \mathscr{A}$ is defined as

$$\Lambda^{(2)}_{(p,q)-\varepsilon}(a) = \left\{\lambda \in \mathbb{C} : (a-\lambda)^{(2)}_{p,q} \text{ does not exist (or)} \left\|(a-\lambda)^{(2)}_{p,q}\right\| \geq \varepsilon\right\}.$$

In the following is an example, we find the $(p,q)-\varepsilon$ pseudo spectrum explicitly.

Example 3 Consider the Banach algebra $B(\mathbb{C}^n)$ where $\mathbb{C}^n$ is the Euclidean space. Let $T \in B(\mathbb{C}^n)$ such that $T(e_i) = \alpha_i e_i$ for some $\alpha_i \in \mathbb{C}$ and the projection operator $P \in B(\mathbb{C}^n)$ defined as $P(e_1) = e_1$ and $P(e_i) = 0$ for all $i = 2$ to n. For any $\lambda \in \mathbb{C} \setminus \{\alpha_1\}$, we define the operator $S(\lambda) \in B(\mathbb{C}^n)$ by

$$[S(\lambda)](e_i) = \begin{cases} \frac{1}{\alpha_1-\lambda} e_1 \text{ for } i = 1 \\ 0 \text{ otherwise }. \end{cases}$$

It is easy to see, $R^{(2)}_{(P,I-P)}(\lambda) = S(\lambda)$ for any $\lambda \in \mathbb{C} \setminus \{\alpha_1\}$. Hence

$$\Lambda^{(2)}_{(P,I-P)-\varepsilon}(T) = \left\{\lambda \in \mathbb{C} : |\lambda - \alpha_1| \leq \frac{1}{\varepsilon}\right\}.$$

Theorem 3 *The set $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$ is a compact subset of $\mathbb{C}$.*

Proof *We know, $\Lambda^{(2)}_{(p,q)-\varepsilon}(a) = \sigma^{(2)}_{p,q}(a) \cup \{\lambda \in \mathbb{C} \mid \|R_a(\lambda)\| \geq \varepsilon\}$. By Theorem 1, $\sigma^{(2)}_{p,q}(a)$ is closed. The set $\{\lambda \in \mathbb{C} : \|R_a(\lambda)\| \geq \varepsilon\}$ is closed, because the map $\lambda \mapsto \|R_a(\lambda)\|$ is continuous. By lemma 1, for any $\lambda \in \rho(a) \cap \Lambda^{(2)}_{(p,q)-\varepsilon}(a)$ with $|\lambda| > \|a\|$, we have*

$$\varepsilon \leq \|R_a(\lambda)\| = \|p(a-\lambda)^{-1}\| \leq \|p\| \frac{1}{|\lambda| - \|a\|}.$$

The above equation implies, $|\lambda| \leq \frac{\|p\|}{\varepsilon} + \|a\|$. Hence $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$ is compact.

Theorem 4 *The set $\sigma^{(2)}_{p,q}(a)$ is a nonempty subset of $\mathbb{C}$. In particular, $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$ is a nonempty subset of $\mathbb{C}$.*

Proof *Suppose* $\sigma^{(2)}_{p,q}(a)=\emptyset$ *then* $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)=\left\{\lambda\in\rho^{(2)}_{p,q}(a):\|R_a(\lambda)\|\geq\varepsilon\right\}$. *Since* $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$ *is compact, there exists* $M>0$ *such that* $\|R_a(\lambda)\|\leq M$ *for all* $\lambda\in\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$. *Consequently,*

$$\|R_a(\lambda)\|\leq M \text{ for every } \lambda\in\mathbb{C}. \tag{3}$$

Since $\rho^{(2)}_{p,q}(a)=\mathbb{C}$ *and the map* $\lambda\mapsto R_a(\lambda)$ *is analytic and bounded on* $\mathbb{C}$, *by Theorem 19.1 in [1], there exists a constant K such that*

$$\|R_a(\lambda)\|\equiv K \text{ for all } \lambda\in\mathbb{C}.$$

If $K=0$ *then* $R_a(\lambda)=0$, *this implies* $p=0$, *which is a contradiction to our assumption* $p\neq 0$. *If* $K>0$ *then* $\Lambda^{(2)}_{(p,q)-K}(a)$ *is unbounded, which is a contradiction to Theorem 3. Hence* $\sigma^{(2)}_{p,q}(a)\neq\emptyset$. *By definition 4,* $\sigma^{(2)}_{p,q}(a)\subseteq\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$. *Thus* $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)\neq\emptyset$.

Theorem 5 *Let* $a\in\mathscr{A}$ *and* $\varepsilon>0$. *Then* $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$ *has no isolated points.*

Proof *Every point in* $\sigma^{(2)}_{(p,q)}(a)$ *is an interior point of* $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$. *Otherwise, there exists a sequence* $\{\lambda_n\}$ *with* $\lambda_n\in\rho^{(2)}_{(p,q)}(a)$ *and* $\|R_a(\lambda_n)\|<\varepsilon$ *such that* $\lambda_n\to\lambda$. *This is a contradiction to corollary 1. Since the map* $\lambda\mapsto\|R_a(\lambda)\|$ *is continuous, the set*

$$\left\{\lambda\in\rho^{(2)}_{(p,q)}(a):\|R_a(\lambda)\|>\varepsilon\right\} \tag{4}$$

is open and hence every λ *which satisfies* $\|R_a(\lambda)\|>\varepsilon$ *is an interior point of* $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$. *Next, we consider a point* $\mu\in\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$ *such that* $\|R_a(\mu)\|=\varepsilon$. *If* μ *is an isolated point then there exists an* $r>0$ *such that* $\|R_a(\lambda)\|<\varepsilon$ *for every* $\lambda\in B(\mu,r)$.*Take* $\Omega_0=\Omega=B(\mu,r)$ *define the following map*

$$F:\Omega_0\to\mathscr{A} \text{ defined by } F(\lambda)=R_a(\lambda).$$

We apply Theorem 2.1 in [13] and it gives us $\|R_a(\mu)\|<\varepsilon$, *which is a contradiction.*

3 Level Sets of (p,q)- Outer Generalized Pseudo Spectrum

This section focuses on $L^{(2)}_{(p,q)-\varepsilon}(a)$. By proving a version of maximum modulus principle (see Theorem 6) to the (p,q) resolvent map, we prove that $L^{(2)}_{(p,q)-\varepsilon}(a)$ has empty interior in the unbounded component of $\rho^{(2)}_{p,q}(a)$. We observed a similar kind of result to any non scalar operator T acting on the complex uniformly convex Banach space X irrespective of the size of component of $\rho^{(2)}_{p,q}(T)$. With the help of these results, we also look at some topological property (see Theorem 8) of $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$.

Note 4 The set $L^{(2)}_{(p,q)-\varepsilon}(a)$ is non empty. Otherwise $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$ is a nonempty open as well as closed subset of $\mathbb{C}$. This is a contradiction to the fact $\mathbb{C}$ is connected.

Note 5 Let μ be a point of the boundary of $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$. By Theorem, 3 $\|R_a(\mu)\| \geq \varepsilon$. Suppose $\|R_a(\mu)\| > \varepsilon$, then by Theorem 5, μ is an interior point of $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$. This is a contradiction to μ is a boundary point. Hence $\mu \in L^{(2)}_{(p,q)-\varepsilon}(a)$. Consequently, boundary set of $\Lambda^{(2)}_{(p,q)-\varepsilon}(a)$ is a subset of $L^{(2)}_{(p,q)-\varepsilon}(a)$.

The following is a form of maximum modulus principle to the map $\lambda \mapsto [R_a(\lambda)]^n$.

Theorem 6 *Let $a \in \mathscr{A}$, Ω be an open subset in the unbounded component of $\rho^{(2)}_{(p,q)}(a)$ and $n \in \mathbb{N}$. For some $M > 0$, suppose $\left\| [(a-\lambda)^n]^{(2)}_{p,q} \right\| \leq M$ for all $\lambda \in \Omega$, then $\left\| [(a-\lambda)^n]^{(2)}_{p,q} \right\| < M$ for all $\lambda \in \Omega$.*

Proof *Let us take the unbounded component of $\rho^{(2)}_{(p,q)}(a)$ be Ω_0. By note 2, for any $n \in \mathbb{N}$, $[(a-\lambda)^n]^{(2)}_{p,q} = [R_a(\lambda)]^n$ for all $\lambda \in \Omega_0$. We note the following,*

$$\{\lambda \in \mathbb{C} : \|[R_a(\lambda)]^n\| \geq M\} \subseteq \left\{\lambda \in \mathbb{C} : \|R_a(\lambda)\| \geq M^{\frac{1}{n}}\right\}.$$

By theorem 3, $\left\{\lambda \in \mathbb{C} : \|R_a(\lambda)\| \geq M^{\frac{1}{n}}\right\}$ is bounded and hence

$$\{\lambda \in \mathbb{C} : \|[R_a(\lambda)]^n\| < M\} \cap \Omega_0 \neq \emptyset.$$

Take $\mu \in \{\lambda \in \mathbb{C} : \|[R_a(\lambda)]^n\| < M\} \cap \Omega_0$. Proof follows by applying theorem 2.1 in [13] to the analytic function $\lambda \mapsto [R_a(\lambda)]^n$ defined from Ω_0 to $\mathscr{A}$, the open set Ω and to the point μ.

Corollary 2 *Let $a \in \mathscr{A}$ and $\varepsilon > 0$. Then $L^{(2)}_{(p,q)-\varepsilon}(a)$ has empty interior in the unbounded component of $\rho^{(2)}_{(p,q)}(a)$*

Proof *Follows from Theorem 6, by applying $n = 1$.*

Our next aim is to prove the interior of $L^{(2)}_{(p,q)-\varepsilon}(T)$ is empty in any component of $\rho^{(2)}_{p,q}(T)$ where $T \in B(X)$ and X is complex uniformly convex Banach space. We prove that, if $\|[R_T(\lambda)]^n\|$ is constant in an open set of $\rho^{(2)}_{p,q}(T)$ then it is the global minimum of $\|[R_T(\lambda)]^n\|$ for all $\lambda \in \rho^{(2)}_{p,q}(T)$. The following is the definition of complex uniformly convex Banach space, with the help of lemma 2, we obtain the required result.

Definition 5 **([13], Definition 2.4 (ii))** A complex Banach space X is said to be complex uniformly convex (uniformly convex) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in X,\ \|y\| \geq \varepsilon \text{ and } \|x + \zeta y\| \leq 1,\ \forall \zeta \in \mathbb{C}\,(\zeta \in \mathbb{R}),\ \text{with } |\zeta| \leq 1 \Rightarrow \|x\| \leq 1 - \delta.$$

It is so obvious that every uniformly convex Banach space is complex uniformly convex Banach space and hence L_p (with $1 < p < \infty$) spaces are complex uniformly convex Banach spaces. In [7], Theorem 1, Globvink showed L_1 space is complex uniformly convex. The Banach space L_∞ is not complex uniformly convex Banach space.

Lemma 2 *([9], Lemma 1.1) Let $\lambda \mapsto f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots$ be a function with values in a complex Banach space X, defined and analytic in a neighbourhood of the point 0 in the complex plane. If $\|f(\lambda)\| \equiv \|a_0\|$ in a neighbourhood of the point 0, then for each a_i $(i = 1,2,\cdots)$ an $r_i > 0$ exists such that $\|a_0 + \lambda a_i\| \le \|a_0\|$ $(|\lambda| \le r_i)$.*

Proof of the following theorem goes similar to the proof of the Theorem 3.4 in [2].

Theorem 7 *Let $T \in B(X)$ where X be a complex uniformly convex Banach space and $n \in \mathbb{N}$. If $\left\|[(T-\lambda)^n]^{(2)}_{p,q}\right\| = 1$ in an open subset U of $\rho^{(2)}_{p,q}(T)$ then $\left\|[(T-\lambda)^n]^{(2)}_{p,q}\right\| \ge 1$ for all $\lambda \in \rho^{(2)}_{p,q}(T)$.*

Proof *We know, $[(T-\lambda)^n]^{(2)}_{p,q} = [R_T(\lambda)]^n$. By Theorem 2, for every fixed $\lambda_0 \in U$, there exists an $r > 0$ such that the map,*

$$f : B(0,r) \to B(X) \text{ defined by } f(\lambda) = [R_T(\lambda+\lambda_0)]^n$$

is analytic at 0. Moreover, for any $\lambda \in B(0,r)$,

$$[R_T(\lambda+\lambda_0)]^n = \left[\sum_{i=0}^{\infty} [R_T(\lambda_0)]^{i+1}\lambda^i\right]^n = [R_T(\lambda_0)]^n + n[R_T(\lambda_0)]^{n+1}\lambda + \mathscr{O}(\lambda^2).$$

Take $a_0 = [R_T(\lambda_0)]^n$ and $a_1 = n[R_T(\lambda_0)]^{n+1}$. Since $\|f(\lambda)\| = \|a_0\|$, by lemma 2, there exists $r_1 > 0$ such that

$$\left\|[R_T(\lambda_0)]^n + \lambda n [R_T(\lambda_0)]^{n+1}\right\| \le 1 \text{ for all } |\lambda| \le r_1.$$

Hence for any $\lambda \in B(0,1)$,

$$\left\|[R_T(\lambda_0)]^n + r_1\lambda n [R_T(\lambda_0)]^{n+1}\right\| \le 1. \tag{5}$$

There exists a sequence $\{e_k\}$ from X with $\|e_k\| = 1$, such that

$$\lim_{k\to\infty} \|[R_T(\lambda_0)]^n (e_k)\| = \|[R_T(\lambda_0)]^n\| = 1. \tag{6}$$

Equation (5) implies,

$$\left\|[R_T(\lambda_0)]^n (e_k) + r_1\lambda n [R_T(\lambda_0)]^{n+1}(e_k)\right\| \le 1. \tag{7}$$

Take $x_k = [R_T(\lambda_0)]^n (e_k)$ and $y_k = r_1 n [R_T(\lambda_0)]^{n+1}(e_k)$.

We claim that $\lim_{k\to\infty}\|y_k\| = 0$. *Suppose* $\|y_k\| \geq \varepsilon$ *for some* $\varepsilon > 0$ *then by equation (7),*

$$\|x_k + \lambda y_k\| \leq 1 \text{ for all } \lambda \in B(0,1). \tag{8}$$

From the definition of complex uniformly convex Banach space, there exists $\delta > 0$ *such that*

$$\|x_k\| \leq 1-\delta.$$

This is a contradiction to equation (6). Hence,

$$\lim_{k\to\infty}\|y_k\| = \lim_{k\to\infty}\left\|r_1 n\,[R_T(\lambda_0)]^{n+1}(e_k)\right\| = 0.$$

Therefore,

$$\lim_{k\to\infty}\left\|[R_T(\lambda_0)]^{n+1}(e_k)\right\| = 0. \tag{9}$$

For any $\lambda \in \rho^{(2)}_{p,q}(a)$, *by Theorem 4.2 (a) in [6]*

$$R_T(\lambda) - R_T(\lambda_0) = (\lambda-\lambda_0)\,[R_T(\lambda)]\,[R_T(\lambda_0)] = (\lambda-\lambda_0)\,[I + (\lambda-\lambda_0)\,[R_T(\lambda)]]\,[R_T(\lambda_0)]^2. \tag{10}$$

where I *denotes the identity operator on* $B(X)$. *From equation (10), it is easy to see,*

$$[R_T(\lambda)]^n - [R_T(\lambda_0)]^n = B_n\,[R_T(\lambda_0)]^{n+1} \tag{11}$$

where $B_n := \sum_{j=0}^{n-1}\binom{n}{j+1}(\lambda-\lambda_0)^{j+1}(I + (\lambda-\lambda_0)R_T(\lambda))^{j+1}(R_T(\lambda_0))^j$. *Since the operator* B_n *is bounded and from the equation (6), equation (9),*

$$\begin{aligned}\lim_{k\to\infty}\|[R_T(\lambda)]^n(e_k)\| &\geq \lim_{k\to\infty}\|[R_T(\lambda_0)]^n(e_k)\| - \lim_{k\to\infty}\left\|B_n\,[R_T(\lambda_0)]^{n+1}(e_k)\right\|\\ &\geq \lim_{k\to\infty}\|[R_T(\lambda_0)]^n(e_k)\| - \|B_n\|\lim_{k\to\infty}\left\|[R_T(\lambda_0)]^{n+1}(e_k)\right\|\\ &= 1.\end{aligned}$$

Hence the theorem follows.

Corollary 3 *Let* $M > 0$, $T \in B(X)$ *where* X *be a complex uniformly convex Banach space and* $n \in \mathbb{N}$. *If* $\left\|[(T-\lambda)^n]^{(2)}_{p,q}\right\| = M$ *in an open subset* U *of* $\rho^{(2)}_{p,q}(T)$ *then* $\left\|[(T-\lambda)^n]^{(2)}_{p,q}\right\| \geq M$ *for all* $\lambda \in \rho^{(2)}_{p,q}(T)$.

Proof Suppose $\left\|[(T-\lambda)^n]^{(2)}_{p,q}\right\| = M$ *in an open subset* U *of* $\rho^{(2)}_{p,q}(T)$, *then*

$$\left\|\left[\left(M^{\frac{1}{n}}T - M^{\frac{1}{n}}\lambda\right)^n\right]^{(2)}_{p,q}\right\| = 1 \text{ for all } \lambda \in U. \tag{12}$$

Consider the operator $S := M^{\frac{1}{n}}T$. *From equation (12), for each* $\mu \in M^{\frac{1}{n}}U$, *we obtain* $\left\|[(S-\mu)^n]^{(2)}_{p,q}\right\| = 1$. *By Theorem 7,* $\left\|[(S-\mu)^n]^{(2)}_{p,q}\right\| \geq 1$ *for all* $\mu \in \rho^{(2)}_{p,q}(S)$. *Thus* $\left\|[(T-\lambda)^n]^{(2)}_{p,q}\right\| \geq M$ *for all* $\lambda \in \rho^{(2)}_{p,q}(T)$.

Corollary 4 *Let X be a complex uniformly convex Banach space. If $T \in B(X)$ then $L^{(2)}_{(p,q)-\varepsilon}(T)$ has empty interior in $\rho^{(2)}_{(p,q)}(T)$.*

Proof *Immediate from corollary 3 by applying $n=1$.*

Theorem 8 *Let X be a complex uniformly convex Banach space, $T \in B(X)$. Then $\Lambda^{(2)}_{(p,q)-\varepsilon}(T)$ has finite number of components and every component of $\Lambda^{(2)}_{(p,q)-\varepsilon}(T)$ contains an element from $\sigma^{(2)}_{p,q}(T)$.*

Proof *Let E be a component of $\Lambda^{(2)}_{(p,q)-\varepsilon}(T)$. We first prove the following,*

$$\text{if } E \cap \{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \varepsilon\} \neq \emptyset \text{ then } E \cap \sigma^{(2)}_{p,q}(T) \neq \emptyset.$$

Assume to the contrary that E is a component and $E \cap \{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \varepsilon\} \neq \emptyset$ but $E \cap \sigma^{(2)}_{p,q}(T) = \emptyset$. Consider the set

$$G := E \setminus \left(L^{(2)}_{(p,q)-\varepsilon}(T)\right) = E \cap \left(L^{(2)}_{(p,q)-\varepsilon}(T)\right)^c.$$

Note that, $G \subseteq \{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \varepsilon\} \subseteq \left(L^{(2)}_{(p,q)-\varepsilon}(T)\right)^c$. We prove that G is open in $\mathbb{C}$. Let $\mu \in G$. Since $\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \varepsilon\}$ is open, there exists $r_\mu > 0$ such that

$$B\left(\mu, r_\mu\right) \subseteq \{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \varepsilon\} \subseteq \left(L^{(2)}_{(p,q)-\varepsilon}(T)\right)^c$$

Since E is a component, $\mu \in E$ and $B\left(\mu, r_\mu\right)$ is connected, we have $B\left(\mu, r_\mu\right) \subseteq E$. By the definition of G, $B\left(\mu, r_\mu\right) \subseteq G$, it follows that G is open in $\mathbb{C}$. Let $\mu \in G$, hence there exists $F \in B(X)^*$ such that $F\left(R_T(\mu)\right) = \|R_T(\mu)\|$. Define

$$\psi : G \to \mathbb{C} \text{ by } \psi(\lambda) = F\left(R_T(\lambda)\right).$$

Clearly ψ is well defined, analytic and also continuous on $\overline{G}$ (closure of G). For any boundary point λ of G we have $\|R_T(\lambda)\| = \varepsilon$, hence $|\psi(\lambda)| \leq \varepsilon$ but at the point μ, we have $|\psi(\mu)| = |F\left(R_T(\mu)\right)| = \|R_T(\mu)\| > \varepsilon$. This is a contradiction to Maximum Modulus Theorem.

By corollary 1, for each $\lambda \in \sigma^{(2)}_{p,q}(T)$, there exists $r_\lambda > 0$ with $B(\lambda, r_\lambda) \subseteq \Lambda^{(2)}_{(p,q)-\varepsilon}(T)$ and $\left\{B(\lambda, r_\lambda) : \lambda \in \sigma^{(2)}_{p,q}(T)\right\}$ is an open cover for $\sigma^{(2)}_{p,q}(T)$. Since $\sigma^{(2)}_{p,q}(T)$ is compact, there exists $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ such that $\sigma^{(2)}_{p,q}(T) \subseteq \bigcup_{i=1}^{n} B\left(\lambda_i, r_{\lambda_i}\right)$. Consequently, there exists components $C_1, C_2, \cdots, C_m$ of $\Lambda^{(2)}_{(p,q)-\varepsilon}(T)$ with $m \leq n$ and each C_i contains atleast one $B\left(\lambda_i, r_{\lambda_i}\right)$ such that

$$\sigma^{(2)}_{p,q}(T) \subseteq \bigcup_{i=1}^{n} B\left(\lambda_i, r_{\lambda_i}\right) \subseteq \bigcup_{i=1}^{m} C_i.$$

We claim that $\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \varepsilon\} \subseteq \bigcup_{i=1}^{m} C_i$. *For* $\mu \in \{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \varepsilon\}$, *there exists* $r > 0$ *such that* $B(\mu, r) \subseteq \{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \varepsilon\}$ *hence* $B(\mu, r) \subseteq E$ *for some connected component* E *of* $\Lambda^{(2)}_{(p,q)-\varepsilon}(T)$. *We proved that* $E \cap \sigma^{(2)}_{p,q}(T) \neq \emptyset$, *it follows that* $E \subseteq \bigcup_{i=1}^{m} C_i$. *Thus*

$$\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \varepsilon\} \subseteq \bigcup_{i=1}^{m} C_i.$$

Since each C_i *is closed in* $\mathbb{C}$ *and by Theorem 5, corollary 4, we have*

$$\overline{\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \varepsilon\}} = \Lambda^{(2)}_{(p,q)-\varepsilon}(T) = \bigcup_{i=1}^{m} C_i.$$

Hence the theorem follows.

The following is an example for interior of $L^{(2)}_{(p,q)-\varepsilon}(a)$ can be nonempty in the bounded component of $\rho^{(2)}_{p,q}(a)$.

Example 4 Consider the Banach space $\ell_\infty(\mathbb{Z})$ with norm

$$\|x\|_* = |x_0| + \sup_{n \neq 0} |x_n| \text{ where } x = \left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right),$$

and the box represents the zeroth coordinate of an element in $\ell_\infty(\mathbb{Z})$. For $M > 2$, take an operator $A \in B(\ell_\infty(\mathbb{Z}))$ such that

$$A\left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right) = \left(\cdots, x_{-2}, x_{-1}, x_0, \boxed{\frac{x_1}{M}}, x_2, x_3, \cdots\right). \tag{13}$$

Take $R := \min\left\{\frac{1}{M}, \frac{1}{2} - \frac{1}{M}\right\}$ and from Theorem 3.1 in [13], we know that

$$\left\|(A-\lambda)^{-1}\right\| = \left\|(A-\lambda)^{-1}(e_0)\right\|_* = M \tag{14}$$

where $e_0 = \left(\cdots, 0, 0, \boxed{1}, 0, 0, \cdots\right)$ and $\lambda \in \mathbb{C}$ such that $|\lambda| < R$. Consider the Banach space $X = \ell_\infty(\mathbb{Z}) \oplus \ell_\infty(\mathbb{Z})$ with norm $\|(x,y)\| = \left(\|x\|_*^2 + \|y\|_*^2\right)^{\frac{1}{2}}$. By Theorem 1.8.6 in [12], X is a Banach space. We take the following operators

$$T : X \to X \text{ defined by } T(x,y) = (A(x), A(y))$$

where A is an operator defined in equation (13) and

$$P : X \to X \text{ defined by } P(x,y) = (x, 0).$$

It is easy to see that $P^2 = P$ and $PT = TP$. By Theorem 1.8.12 in [12], $\sigma(T) = \sigma(A)$ and so we get,

$$R_T(\lambda) = (T-\lambda)^{-1} P \text{ for all } \lambda \in \{\lambda \in \mathbb{C} : |\lambda| < R\}$$

For any $(x,y) \in X$ with $\|(x,y)\| = 1$, we have

$$\left\|(T-\lambda)^{-1} P(x,y)\right\| = \left\|(T-\lambda)^{-1}(x,0)\right\| = \left\|(A-\lambda)^{-1}(x)\right\|_* \leq M\|x\|_* \leq M\|(x,y)\|. \tag{15}$$

and particularly for the unit vector $(e_0,0) \in X$, we have

$$\left\|(T-\lambda)^{-1} P(e_0,0)\right\| = \left\|(T-\lambda)^{-1}(e_0,0)\right\| = \left\|(A-\lambda)^{-1}(e_0)\right\|_* = M = M\|(e_0,0)\|. \tag{16}$$

From equation (15) and equation (16), we get $\left\|(T-\lambda)^{-1} P\right\| = M$ for each λ in $\{\lambda \in \mathbb{C} : |\lambda| < R\}$. Thus interior of $\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| = M\}$ is non empty.

4 Compliance with Ethical Standards

Conflict of Interest: The authors have equally contributed and give their consent for publication.The authors declare that they have no conflict of interest.

Research involving human participants and/or animals: This paper does not contain any studies involving with human participants/ animals.

References

[1] G. Bachman and L. Narici, Functional analysis, Academic Press, New York, 1966.
[2] S. Bögli and P. Siegl, Remarks on the convergence of pseudospectra, Integral Equations Operator Theory **80**, no. 3. (2014), 303–321.
[3] A. Böttcher, S. M. Grudsky and B. Silbermann, Norms of inverses, spectra, and pseudospectra of large truncated Wiener-Hopf operators and Toeplitz matrices, New York J. Math. **3** (1997), 1–31.
[4] A. Böttcher, Pseudospectra and singular values of large convolution operators, J. Integral Equations Appl. **6** (1994), 267–301.
[5] A. Böttcher and S. M. Grudsky, Spectral properties of banded Toeplitz matrices, SIAM, Philadelphia, PA, 2005.
[6] D. S. Djordjević and Y. Wei, Outer generalized inverses in rings, Comm. Algebra **33**, no. 9. (2005), 3051–3060.
[7] J. Globevnik, On complex strict and uniform convexity, Proc. Amer. Math. Soc. **47** (1975), 175–178.
[8] J. Globevnik, Norm-constant analytic functions and equivalent norms, Illinois J. Math. **20**, no. 3. (1976), 503–506.
[9] J. Globevnik and I. Vidav, On operator-valued analytic functions with constant norm, J. Functional Analysis **15** (1974), 394–403.
[10] M. Z. Kolundžija, (P,Q)-outer generalized inverse of block matrices in Banach algebras, Banach J. Math. Anal. **8**, no. 1. (2014), 98–108.
[11] J. Locker, Spectral theory of non-self-adjoint two-point differential operators, Mathematical Surveys and Monographs, 73, Amer. Math. Soc., Providence, RI, 2000.
[12] R. E. Megginson, An introduction to Banach space theory, Graduate Texts in Mathematics, 183, Springer, New York, 1998.
[13] E. Shargorodsky, On the level sets of the resolvent norm of a linear operator, Bull. Lond. Math. Soc. **40**, no. 3. (2008), 493–504.