Chordal Bipartite Graphs with High Boxicity

L. Sunil Chandran, Mathew C. Francis, and Rogers Mathew

Department of Computer Science and Automation, Indian Institute of Science,
Bangalore – 560012, India.
{sunil,mathew,rogers}@csa.iisc.ernet.in

Abstract. The boxicity of a graph G is defined as the minimum integer k such that G is an intersection graph of axis-parallel k-dimensional boxes. Chordal bipartite graphs are bipartite graphs that do not contain an induced cycle of length greater than 4. It was conjectured by Otachi, Okamoto and Yamazaki that chordal bipartite graphs have boxicity at most 2. We disprove this conjecture by exhibiting an infinite family of chordal bipartite graphs that have unbounded boxicity.

Key words: Boxicity, chordal bipartite graphs, interval graphs, grid intersection graphs.

1 Introduction

A graph G is an *intersection graph* of sets from a family of sets $\mathcal{F}$, if there exists $f : V(G) \rightarrow \mathcal{F}$ such that $(u, v) \in E(G) \Leftrightarrow f(u) \cap f(v) \neq \emptyset$. An *interval graph* is an intersection graph in which the set assigned to each vertex is a closed interval on the real line. In other words, interval graphs are intersection graphs of closed intervals on the real line. An *axis-parallel k-dimensional box* in $\mathbb{R}^k$ is the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$, where each R_i is an interval of the form $[a_i, b_i]$ on the real line. *Boxicity* of any graph G (denoted by $\mathrm{box}(G)$) is the minimum integer k such that G is an intersection graph of axis-parallel k-dimensional boxes in $\mathbb{R}^k$. Note that interval graphs are exactly those graphs with boxicity at most 1.

The concept of boxicity was introduced by F. S. Roberts in the year 1969 [13]. It finds applications in niche overlap (competition) in ecology and to problems of fleet maintenance in operations research (see [8]). Roberts proved that the boxicity of any graph on n vertices is upper bounded by $\lfloor \frac{n}{2} \rfloor$. He also showed that a complete $\frac{n}{2}$-partite graph with 2 vertices in each part has its boxicity equal to $\frac{n}{2}$. Various other upper bounds on boxicity in terms of graph parameters such as maximum degree and treewidth were proved by Chandran, Francis and Sivadasan. In [4] they showed that, for any graph G on n vertices having maximum degree Δ, $\mathrm{box}(G) \leq (\Delta + 2) \ln n$. They also upper bounded boxicity solely in terms of the maximum degree Δ of a graph by showing that $\mathrm{box}(G) \leq 2\Delta^2$ [5]. This means that the boxicity of degree bounded graphs do not is bounded no matter what the size of the vertex set is. It was shown in [6] by Chandran and Sivadasan that $\mathrm{box}(G) \leq \mathrm{tw}(G) + 2$, where $\mathrm{tw}(G)$ denotes the treewidth of graph G.

Cozzens [7] proved that given a graph, the problem of computing its boxicity is NP-hard. Several attempts have been made to find good upper bounds for the boxicity of special classes of graphs. It was shown by Thomassen in [15] that planar graphs have boxicity at most 3. Meanwhile, Scheinerman [14] proved that outerplanar graphs have boxicity at most 2. The boxicity of split graphs was investigated by Cozzens and Roberts [8]. Apart from these results, not much is known about the boxicity of most of the well-known graph classes.

1.1 Chordal Bipartite Graphs (CBGs)

A bipartite graph G is a *chordal bipartite graph* (CBG) if G does not have an induced cycle of length greater than 4. In other words, all induced cycles in such a bipartite graph will be of length exactly equal to 4. Chordal bipartite graphs were introduced by Golumbic and Goss [11], as a natural bipartite analogue of chordal graphs. Chordal bipartite graphs are a well studied class of graphs and several characterizations have been found, such as by the elimination scheme, minimal edge separators, Γ-free matrices etc. (refer [10]).

1.2 Our Result

In 2007, Otachi, Okamoto and Yamazaki [12] proved that P_6-free chordal bipartite graphs have boxicity at most 2. In the same paper, they also conjectured that the boxicity of any chordal bipartite graph is upper bounded by the same constant 2. We disprove this conjecture by showing that there exist chordal bipartite graphs with arbitrarily high boxicity. This result also implies that the class of chordal bipartite graphs is incomparable with the class of "grid intersection graphs" (see [1]).

2 Definitions and Notations

Let $V(G)$ and $E(G)$ denote the vertex set and edge set respectively of a graph G. For any $S \subseteq V(G)$, let $G - S$ denote the graph induced by the vertex set $V(G) \setminus S$ in G. In this paper, we consider only simple, finite, undirected graphs. In a graph G, for any $u \in V(G)$, $N(u)$ denotes its neigbourhood in G, i.e. $N(u) = \{v \mid (u,v) \in E(G)\}$. Also, $N[u]$ denotes the closed neighbourhood of u in G, i.e. $N[u] = N(u) \cup \{u\}$. A graph G is a bipartite graph if there is a partition of $V(G)$ into two sets A and B such that both A and B induce independent sets in G. We call $\{A, B\}$ the *bipartition* of the bipartite graph G. Given a tree T and two vertices u and v in T, we denote by uTv the unique path in T between u and v (including u and v). If G, G_1, $G_2, \ldots, G_k$ are $k+1$ graphs, where $V(G) = V(G_1) = V(G_2) = \cdots = V(G_k)$, then we say that $G = G_1 \cap G_2 \cap \cdots \cap G_k$ if $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_k)$.

2.1 Interval Graphs and Boxicity

Since an interval graph is the intersection graph of closed intervals on the real line, for every interval graph I, there exists a function $f : V(I) \rightarrow \{X \subseteq \mathbb{R} \mid X \text{ is a closed interval}\}$, such that for $u, v \in V(I)$, $(u,v) \in E(I) \Leftrightarrow f(u) \cap f(v) \neq \emptyset$. The function f is called an *interval representation* of the interval graph I. Note that it is possible for an interval graph to have more than one interval representation. Given a closed interval $X = [y,z]$, we define $l(X) := y$ and $r(X) := z$. Also note that by the definition of an interval, if $[y,z]$ is an interval, $y \leq z$. For any two intervals $[y_1, z_1], [y_2, z_2]$ on the real line, we say that $[y_1, z_1] < [y_2, z_2]$ if $z_1 < y_2$. Clearly, $[y_1, z_1] \cap [y_2, z_2] = \emptyset$ if and only if $[y_1, z_1] < [y_2, z_2]$ or $[y_2, z_2] < [y_1, z_1]$.

Let I be an interval graph and f an interval representation of I. Let $y, z \in \mathbb{R}$ with $y \leq z$. Then any set of vertices, say $S = \{u_1, u_2, \ldots, u_k\}$ where $S \subseteq V(I)$ and $k > 0$, is said to "overlap in the region $[y,z]$ in f" if each $f(u_i)$ (where $1 \leq i \leq k$) contains the region $[y,z]$, i.e. for each $u_i \in S$, $l(f(u_i)) \leq y \leq z \leq r(f(u_i))$.

A graph G is *chordal* if it does not contain any induced cycle of length greater than 3. The following is a well known fact about interval graphs.

Lemma 1. *All interval graphs are chordal.*

We have seen that interval graphs are intersection graphs of intervals on the real line. The following lemma gives the relationship between intersection graphs of axis-parallel k-dimensional boxes and interval graphs.

Lemma 2 (Roberts[13]). *For any graph G,* $\mathrm{box}(G) \leq b$ *if and only if there exist b interval graphs $I_1, I_2, \ldots, I_b$, with $V(G) = V(I_1) = V(I_2) = \cdots = V(I_b)$ such that $G = I_1 \cap I_2 \cap \cdots \cap I_b$.*

From the above lemma, we can say that the boxicity of a graph G is the minimum b for which there exist b interval graphs $I_1, \ldots, I_b$ such that $G = I_1 \cap I_2 \cap \cdots \cap I_b$. Note that if $G = I_1 \cap I_2 \cap \cdots \cap I_b$, then each I_i is a supergraph of G and also for every pair of vertices $u, v \in V(G)$ such that $(u,v) \notin E(G)$, $(u,v) \notin E(I_i)$, for some i.

2.2 Strongly Chordal Graphs and Chordal Bipartite Graphs

A chordal graph is *strongly chordal* if it does not contain any induced trampoline (refer [9]). Two vertices u and v in a graph are said to be *compatible* if $N[u] \subseteq N[v]$ or vice versa. Otherwise they are said to be *non-compatible.* A vertex v in a graph G is a *simple vertex* if for any $x, y \in N[v]$, x and y are compatible. An ordering $v_1, \ldots, v_n$ of vertices of a graph G is said to be a *simple elimination ordering* if for each i, the vertex v_i is a simple vertex in the graph induced by the vertices $\{v_i, \ldots, v_n\}$ in G. The following characterization of strongly chordal graphs is from page 78 of [3].

Lemma 3. *A graph is strongly chordal if and only if it admits a simple elimination ordering.*

For a bipartite graph G with bipartition $\{A, B\}$, we denote by $C_A(G)$ ($C_B(G)$) the split graph obtained from G by adding edges between every pair of vertices in A (B). A *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set. Recall that a bipartite graph is chordal bipartite if it does not have any induced cycle of length greater than 4. The following characterization of chordal bipartite graphs appears in [2].

Lemma 4. *Let G be a bipartite graph with bipartition $\{A, B\}$. Then, G is chordal bipartite if and only if $C_A(G)$ is strongly chordal.*

2.3 Bipartite Powers

For any two vertices u, v in a graph G, let $d_G(u, v)$ denote the length of a shortest u-v path in G. Given a bipartite graph G and an odd positive integer k, we define the graph $G^{[k]}$ to be the graph with $V\left(G^{[k]}\right) = V(G)$ and $E\left(G^{[k]}\right) = \{(u, v) \mid u, v \in V(G),\ d_G(u, v) \text{ is odd, and } d_G(u, v) \leq k\}$. The graph $G^{[k]}$ is called the *k-th bipartite power* of the bipartite graph G. It is easy to see that if G is a bipartite graph with the bipartition $\{A, B\}$, then $G^{[k]}$ is also a bipartite graph with the bipartition $\{A, B\}$.

3 Bipartite powers of Trees

Let T be a rooted tree with vertex r being its root. T is therefore a bipartite graph and let $\{A, B\}$ be its bipartition. For any $u, v \in V(T)$, we say $u \preceq v$ in T, if $v \in rTu$. Otherwise, we say $u \npreceq v$. For $u, v \in V(T)$, we define $P(u, v) := \{x \in V(T) \mid u \preceq x \text{ and } v \preceq x\}$. The *least common ancestor* (LCA) of any two vertices $u, v \in V(T)$ in T is that vertex $z \in P(u, v)$ such that $\forall y \in P(u, v), z \preceq y$. Note that if z is the LCA of u and v, then $z \in uTv, z \in uTr$ and $z \in vTr$. We say that a vertex u is *farthest* from a vertex v in T if $\forall w \in V(T)$, $d_T(v, w) \leq d_T(v, u)$. Note that in this case u will be a leaf vertex in T.

Lemma 5. *Let $x \in V(T)$ such that x is a leaf vertex in T. Then, $(T - \{x\})^{[k]} = T^{[k]} - \{x\}$.*

Proof. For ease of notation, let $G = (T - \{x\})^{[k]}$ and $G' = T^{[k]} - \{x\}$. Let $(u, v) \in E(G')$. Since x is a leaf vertex in T, $x \notin uTv$. Therefore, $(u, v) \in E(G)$. Hence, $(u, v) \in E(G') \Rightarrow (u, v) \in E(G)$. Also, $(u, v) \in E(G) \Rightarrow (u, v) \in E(G')$ since G is a subgraph of G'. Therefore, $(u, v) \in E(G) \Leftrightarrow (u, v) \in E(G')$. This proves the lemma. □

Lemma 6. *Let $x \in V(T)$ such that x is farthest from r in T. Assume that $x \in A$. For any odd positive integer k, let $G := C_B(T^{[k]})$. Then, x is a simple vertex in G.*

Proof. We shall prove this by proving that, in G, for any two vertices $u_1, u_2 \in N[x]$, such that $d_T(r, u_1) \geq d_T(r, u_2)$, $N[u_1] \subseteq N[u_2]$. That x is farthest from

r in T implies that $u_2 \neq x$ (note that if $d_T(r, u_1) = d_T(r, u_2)$, u_1 and u_2 are interchangeable). Now, since $N[x] \cap A = \{x\}$, we have $u_2 \in B$. Let $v \in N[u_1]$ in G. If $v \in B$, then $v \in N[u_2]$ (since B induces a clique in G). When $v \notin B$, we split the proof into the two cases given below.
Let w be the LCA of u_1 and u_2 in T.

Case (i). $v \preceq w$ in T

We know that since $u_1, u_2 \in N[x]$ in G, $d_T(u_1, x) \leq k$ and $d_T(u_2, x) \leq k$. It is easy to see that $w \in u_1Tx$ or $w \in u_2Tx$. If $w \in u_1Tx$ then, $d_T(u_1, w) + d_T(w, x) = d_T(u_1, x) \leq k$. Otherwise, $d_T(u_2, w) + d_T(w, x) = d_T(u_2, x) \leq k$. Since $d_T(r, u_1) \geq d_T(r, u_2)$ implies that $d_T(u_1, w) \geq d_T(u_2, w)$, we always have $d_T(u_2, w) + d_T(w, x) \leq k$. We know that $d_T(r, x) \leq d_T(r, w) + d_T(w, x)$ and $d_T(r, v) = d_T(r, w) + d_T(w, v)$. Since $d_T(r, v) \leq d_T(r, x)$ (as x is farthest from r in T), we have $d_T(w, v) \leq d_T(w, x)$. Therefore, $d_T(u_2, w) + d_T(w, v) \leq k$. Hence, $v \in N[u_2]$ in G.

Case (ii). $v \npreceq w$ in T

In this case, it is easy to see that $w \in u_1Tv$ and $w \in u_2Tv$. This implies that $d_T(u_1, v) = d_T(u_1, w) + d_T(w, v)$ and $d_T(u_2, v) = d_T(u_2, w) + d_T(w, v)$. Since $d_T(u_1, v) \leq k$ and $d_T(u_1, w) \geq d_T(u_2, w)$, we can conclude that $d_T(u_2, v) \leq k$. Therefore, $v \in N[u_2]$ in G.

This proves that $N[u_1] \subseteq N[u_2]$. Hence the lemma. □

Theorem 1. *For any odd positive integer k, $T^{[k]}$ is a CBG.*

Proof. Let us prove this by using induction on the number of vertices of T. Let $x \in V(T)$ such that x is farthest from r in T. Assume $x \in A$. Let $G := C_B(T^{[k]})$. Then by Lemma 6, x is a simple vertex in G. Let $G' = G - \{x\}$. Note that $G' = C_B(T^{[k]} - \{x\})$. But from Lemma 5, $T^{[k]} - \{x\} = (T - \{x\})^{[k]}$ which by our induction hypothesis is a CBG. Then, applying Lemma 4, we can say that G' is a strongly chordal graph. Since x is a simple vertex in G and since $G' = G - \{x\}$, by applying Lemma 3 we can say that G is also a strongly chordal graph. Therefore by Lemma 4, $T^{[k]}$ is a CBG. ■

4 Boxicity of CBGs

Lemma 7. *In an interval graph I, let $S \subseteq V(I)$ be a set of vertices that induces a clique. Then for an interval representation f of I, $\exists y, z \in \mathbb{R}$ with $y \leq z$ such that S overlaps in the region $[y, z]$ in f.*

Proof. Proof of the lemma follows directly from the Helly property for intervals on the real line. □

Lemma 8. *Let G be a bipartite graph with bipartition $\{A, B\}$. Let G' be the graph with $V(G') = V(G) = V$ and $E(G') = E(G) \cup \{(u, v) \mid u, v \in A$ and $u \neq v\} \cup \{(u, v) \mid u, v \in B$ and $u \neq v\}$. Then,* $\mathrm{box}(G) \geq \frac{\mathrm{box}(G')}{2}$.

Proof. Let $\text{box}(G) = b$. Then by Lemma 2 we have a set of interval graphs, say $\mathcal{I} = \{I_1, I_2, \ldots, I_b\}$ with $V(I_1) = \cdots = V(I_b) = V$, such that $G = I_1 \cap I_2 \cap \cdots \cap I_b$. As each I_i is an interval graph, there exists an interval representation f_i for it. For each i, let $s_i = \min_{x \in V} l(f_i(x))$ and $t_i = \max_{x \in V} r(f_i(x))$. Corresponding to each interval graph I_i in $\mathcal{I}$, we construct two interval graphs I'_i and I''_i. We construct the interval representations f'_i and f''_i for I'_i and I''_irespectively from f_i as follows:

Construction of f'_i:

$\forall u \in A,\ f'_i(u) = [s_i, r(f_i(u))]$.
$\forall u \in B,\ f'_i(u) = [l(f_i(u)), t_i]$.

Construction of f''_i:

$\forall u \in A,\ f''_i(u) = [l(f_i(u)), t_i]$.
$\forall u \in B,\ f''_i(u) = [s_i, r(f_i(u))]$.

Let $\mathcal{I}' = \{I'_1, \ldots, I'_b, I''_1, \ldots, I''_b\}$. Now we claim that $G' = \bigcap_{I \in \mathcal{I}'} I$.

Let $i \in \mathbb{N}$ such that $1 \leq i \leq b$. Since A overlaps in the region $[s_i, s_i]$ in f'_i, the vertices in A induce a clique in I'_i. Similarly, B overlaps in the region $[t_i, t_i]$ in f'_i and hence the vertices in B also induce a clique in I'_i. Also, for any $u \in A$, $v \in B$, $(u, v) \in E(G') \Rightarrow (u, v) \in E(G) \Rightarrow (u, v) \in E(I_i) \Rightarrow (u, v) \in E(I'_i)$ (since $\forall u \in V$, $f_i(u) \subseteq f'_i(u)$). Hence each I'_i is a supergraph of G'. It can be shown by proceeding along similar lines that each I''_i is a supergraph of G'. Therefore, each $I \in \mathcal{I}'$ is a supergraph of G'.

Let $u, v \in V$ such that $(u, v) \notin E(G')$. Both u and v together cannot be in A or B since both A and B induce cliques in G'. Assume without loss of generality that $u \in A$ and $v \in B$. Now, $(u, v) \notin E(G') \Rightarrow (u, v) \notin E(G)$ (follows from the way G' is constructed from G). Since $(u, v) \notin E(G)$, $\exists I_i \in \mathcal{I}$ such that $(u, v) \notin E(I_i)$, i.e. $f_i(u) \cap f_i(v) = \emptyset$. If $f_i(u) < f_i(v)$ then $f'_i(u) < f'_i(v)$ and therefore $(u, v) \notin E(I'_i)$. Otherwise, if $f_i(v) < f_i(u)$ then $f''_i(v) < f''_i(u)$ and therefore $(u, v) \notin E(I''_i)$. To summarise, for any $(u, v) \notin E(G')$, $\exists I \in \mathcal{I}'$ such that $(u, v) \notin E(I)$.

Hence we prove the claim that $G' = \bigcap_{I \in \mathcal{I}'} I$. By Lemma 2, this means that $\text{box}(G') \leq 2b = 2 \cdot \text{box}(G)$. □

Let T_k be the tree shown in figure 1. Here $k \in \mathbb{N}$ is an odd number and $g(k) = \frac{k+1}{2} \cdot (g(k-2) - 1) + 1$ with $g(1) = 2$. Let $G_k = T_k^{[k]}$. It follows from Theorem 1 that G_k is a CBG. Let $L_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,g(k)}\}$ denote the set of all vertices in layer i of T_k. Note that T_k, and consequently G_k, is a bipartite graph with the bipartition $\{A, B\}$ where $A = \{u \in L_i \mid 0 \leq i \leq k+1,\ i \text{ is an odd number}\}$ and $B = \{u \in L_i \mid 0 \leq i \leq k+1,\ i \text{ is an even number}\}$.

Let G'_k be the graph with $V(G'_k) = V(G_k) = V(T_k)$ and $E(G'_k) = E(G_k) \cup \{(u, w) \mid u, w \in A \text{ and } u \neq w\} \cup \{(u, w) \mid u, w \in B \text{ and } u \neq w\}$. So in G'_k, A and B induce cliques.

Lemma 9. $\text{box}(G'_k) > \frac{k+1}{2}$

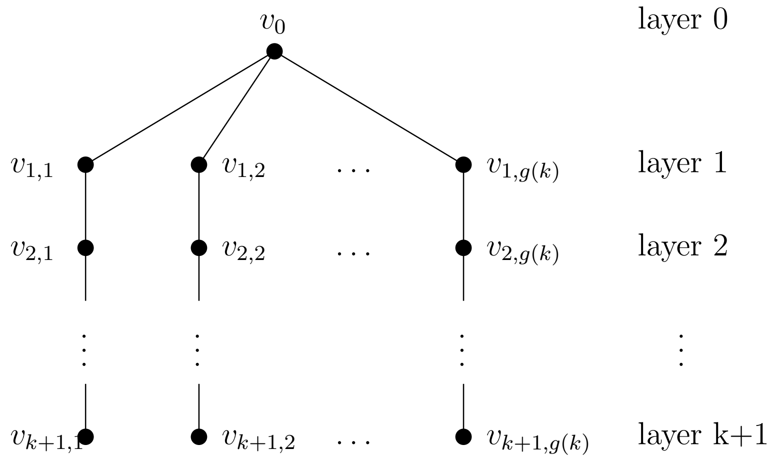

Fig. 1. Tree T_k

Proof. Let $X'_k := G'_k - \{v_0\}$. Since X'_k is an induced subgraph of G'_k, $\mathrm{box}(G'_k) \geq \mathrm{box}(X'_k)$. Now, we prove that $\mathrm{box}(X'_k) > \frac{k+1}{2}$ by using induction on k. Since X'_1 is precisely a cycle of length 4, it is not a chordal graph. Therefore, by Lemma 1, $\mathrm{box}(X'_1) > 1$ and thus our induction hypothesis holds for the case $k = 1$. Let $m \in \mathbb{N}$ be an odd number. Let us assume that our claim is true when k is an odd number and $k < m$. Now, when $k = m$, we need to prove that $\mathrm{box}(X'_m) > \frac{m+1}{2}$. We prove this by contradiction. Assume $\mathrm{box}(X'_m) \leq r = \frac{m+1}{2}$. Then by Lemma 2, there exists a set of r interval graphs, say $\mathcal{I} = \{I_1, I_2, \ldots, I_r\}$, such that $X'_m = I_1 \cap I_2 \cap \cdots \cap I_r$. As each I_i is an interval graph, there exists an interval representation f_i for it.

Since the vertices of L_1 induce a clique in X'_m, they also induce a clique in each $I_i \in \mathcal{I}$. Let $[s_i, t_i] = \bigcap_{u \in L_1} f_i(u)$. Lemma 7 guarantees that $[s_i, t_i] \neq \emptyset$. We know that for each $v_{m+1,p} \in L_{m+1}$, there exists $v_{1,q} \in L_1$ with $p \neq q$ such that $(v_{m+1,p}, v_{1,q}) \notin E(X'_m)$. So for each $v_{m+1,p} \in L_{m+1}$, there exists some $I_j \in \mathcal{I}$ such that $(v_{m+1,p}, v_{1,q}) \notin E(I_j)$. Therefore, $f_j(v_{m+1,p}) \cap [s_j, t_j] = \emptyset$.

Now let us partition L_{m+1} into r sets $P_1, P_2, \ldots, P_r$ such that $P_i = \{u \in L_{m+1} \mid f_i(u) \cap [s_i, t_i] = \emptyset$ and for any $j < i$, $f_j(u) \cap [s_j, t_j] \neq \emptyset\}$. Since $|L_{m+1}| = g(m) = r \cdot (g(m-2) - 1) + 1$, there exists some P_j such that $|P_j| \geq g(m-2)$. Assume $P_j = P_1$. Without loss of generality, let us also assume that $v_{m+1,1}, v_{m+1,2}, \ldots, v_{m+1,g(m-2)} \in P_1$. So $f_1(v_{m+1,1}) \cap [s_1, t_1] = \emptyset$, $f_1(v_{m+1,2}) \cap [s_1, t_1] = \emptyset$, $\ldots$, $f_1(v_{m+1,g(m-2)}) \cap [s_1, t_1] = \emptyset$. Since $X'_m = I_1 \cap I_2 \cap \cdots \cap I_r$, both the sets A and B which induce cliques in X'_m also induce cliques in I_1. Let $[y_A, z_A] = \bigcap_{u \in A} f_1(u)$ and $[y_B, z_B] = \bigcap_{u \in B} f_1(u)$. By Lemma 7, $[y_A, z_A] \neq \emptyset$ and $[y_B, z_B] \neq \emptyset$. Since I_1 is not a complete graph (recall that $f_1(v_{m+1,1}) \cap [s_1, t_1] = \emptyset$ and therefore $v_{m+1,1}$ is not adjacent to some vertex in L_1 in I_1), $[y_A, z_A] \cap [y_B, z_B] = \emptyset$ implying that either $z_A < y_B$ or $z_B < y_A$. Assume $z_A < y_B$ (the proof is similar even otherwise). Therefore we have,

$$y_A \leq z_A < y_B \leq z_B. \tag{1}$$

Since $L_1 \subseteq A$, we have

$$s_1 \leq y_A \leq z_A \leq t_1. \tag{2}$$

For any $1 \leq i \leq m+1$, let $L'_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,g(m-2)}\}$. Since $L'_{m+1} \subseteq L_{m+1} \subseteq B$, for any $u \in L'_{m+1}$,

$$l(f_1(u)) \leq y_B \leq z_B \leq r(f_1(u)). \tag{3}$$

From inequalities 1, 2 and 3, we can see that for any $u \in L'_{m+1}$,

$$s_1 < r(f_1(u)). \tag{4}$$

Note that $L'_{m+1} \subseteq P_1$ and hence for any $u \in L'_{m+1}$, $f_1(u) \cap [s_1, t_1] = \emptyset$, i.e. either

$$t_1 < l(f_1(u)) \text{ or } r(f_1(u)) < s_1$$

is true. From inequality 4, we then conclude that for any $u \in L'_{m+1}$,

$$t_1 < l(f_1(u)). \tag{5}$$

Let $l_{min} = \min_{u \in L'_{m+1}} l(f_1(u))$. Combining inequalities 2, 3 and 5, we have

$$s_1 \leq y_A \leq z_A \leq t_1 < l_{min} \leq y_B \leq z_B. \tag{6}$$

Let $A' = \{u \in L'_i \mid 2 \leq i \leq m \text{ and } i \text{ is an odd number}\}$ and $B' = \{u \in L'_i \mid 2 \leq i \leq m \text{ and } i \text{ is an even number}\}$. Clearly, $A' \subseteq A$ ($B' \subseteq B$) and therefore it overlaps in the region $[y_A, z_A]$ ($[y_B, z_B]$) in f_1. For any $v_{i,j} \in A'$, since $(v_{i,j}, v_{m+1,j}) \in E(X'_m) \subseteq E(I_1)$,

$$l(f_1(v_{i,j})) \leq y_A \leq z_A \leq t_1 < l_{min} \leq l(f_1(v_{m+1,j})) \leq r(f_1(v_{i,j})). \tag{7}$$

Also, since for any $u \in B', w \in L_1$, $(u, w) \in E(X'_m) \subseteq E(I_1)$, we have for any $u \in B'$, $f_1(u) \cap [s_1, t_1] \neq \emptyset$ and therefore,

$$l(f_1(u)) \leq t_1 < l_{min} \leq y_B \leq z_B \leq r(f_1(u)). \tag{8}$$

From inequalities 7 and 8, we can say that $A' \cup B'$ overlaps in the region $[t_1, t_1]$ in f_1. Hence in I_1, $A' \cup B'$ induces a clique. Now, we claim that the graph induced by $A' \cup B'$ in X'_m, say Z, is isomorphic to X'_{m-2}. Let $V(X'_{m-2}) = \{\overline{v}_{1,1}, \ldots, \overline{v}_{1,g(m-2)}, \overline{v}_{2,1}, \ldots, \overline{v}_{2,g(m-2)}, \ldots, \overline{v}_{m-1,1}, \ldots, \overline{v}_{m-1,g(m-2)}\}$. Then it can be easily verified that this isomorphism is given by the mapping $h : V(Z) \to V(X'_{m-2})$ where, for any $v_{i,j} \in V(Z)$, $h(v_{i,j}) = \overline{v}_{i-1,j}$. Since $X'_m = I_1 \cap \cdots \cap I_r$, Z is the graph induced by $A' \cup B'$ in $I_1 \cap \cdots \cap I_r$. But, we have showed that $A' \cup B'$ induces a clique in I_1 which means that $(u, w) \in E(Z) \Leftrightarrow (u, w) \in E(I_2 \cap \cdots \cap I_r)$. Hence, Z is the graph induced by $A' \cup B'$ in $I_2 \cap \cdots \cap I_r$. Therefore, $\frac{m-1}{2} = r - 1 \geq \text{box}(I_2 \cap \cdots \cap I_r) \geq \text{box}(Z) = \text{box}(X'_{m-2})$. But this contradicts our induction hypothesis that $\text{box}(X'_{m-2}) > \frac{m-1}{2}$. Hence we prove the lemma. $\square$

From Lemma 8 and Lemma 9, we get the following lemma.

Lemma 10. $\mathrm{box}(G_k) > \frac{k+1}{4}$.

Theorem 2. *For any* $b \in \mathbb{N}^+$*, there exists a CBG* G *with* $\mathrm{box}(G) > b$*.*

Proof. For any odd positive integer k, since G_k is the bipartite power of a tree T_k, G_k is a CBG by Theorem 1. Let $G = G_{(4b-1)}$. Then by Lemma 10, $\mathrm{box}(G) > b$. ∎

References

1. S. Bellantoni, I. Ben-Arroyo Hartman, T. Przytycka, and S. Whitesides. Grid intersection graphs and boxicity. *Discrete mathematics*, 114(1-3):41–49, April 1993.
2. Andreas Brandstädt. Classes of bipartite graphs related to chordal graphs. *Discrete Applied Mathematics*, 32(1):51–60, June 1991.
3. Andreas Brandstädt, V. Bang Le, and Jeremy P. Spinrad. *Graph Classes: A Survey*. SIAM, Philadelphia, – edition, 1999.
4. L. Sunil Chandran, Mathew C. Francis, and Naveen Sivadasan. Geometric representation of graphs in low dimension. To appear in Algorithmica, available at http://arxiv.org/abs/cs.DM/0605013.
5. L. Sunil Chandran, Mathew C. Francis, and Naveen Sivadasan. Boxicity and maximum degree. *Journal of Combinatorial Theory, Series B*, 98(2):443–445, March 2008.
6. L. Sunil Chandran and Naveen Sivadasan. Boxicity and treewidth. *Journal of Combinatorial Theory, Series B*, 97(5):733–744, September 2007.
7. M. B. Cozzens. Higher and multidimensional analogues of interval graphs. Ph. D. thesis, Rutgers University, New Brunswick, NJ, 1981.
8. M. B. Cozzens and F. S. Roberts. Computing the boxicity of a graph by covering its complement by cointerval graphs. *Discrete Applied Mathematics*, 6:217–228, 1983.
9. Martin Farber. Characterizations of strongly chordal graphs. *Discrete Mathematics*, 43(2-3):173–189, 1983.
10. Martin C Golumbic. *Algorithmic Graph Theory And Perfect Graphs*. Academic Press, New York, 1980.
11. M.C. Golumbic and C.F. Goss. Perfect elimination and chordal bipartite graphs. *J. Graph Theory 2*, pages 155–163, 1978.
12. Yota Otachi, Yoshio Okamoto, and Koichi Yamazaki. Relationships between the class of unit grid intersection graphs and other classes of bipartite graphs. *Discrete Applied Mathematics*, 155:2383–2390, October 2007.
13. F. S. Roberts. *Recent Progresses in Combinatorics*, chapter On the boxicity and cubicity of a graph, pages 301–310. Academic Press, New York, 1969.
14. E. R. Scheinerman. Intersection classes and multiple intersection parameters. Ph. D. thesis, Princeton University, 1984.
15. C. Thomassen. Interval representations of planar graphs. *Journal of Combinatorial Theory, Series B*, 40:9–20, 1986.