A REPRESENTATION OF HYPONORMAL ABSOLUTELY NORM ATTAINING OPERATORS

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Abstract. In this article, we characterize absolutely norm attaining normal operators in terms of the essential spectrum. Later we prove a structure theorem for hyponormal absolutely norm attaining (or $\mathcal{AN}$-operators in short) and deduce conditions for the normality of the operator.

1. Introduction

The class of hyponormal operators is an important class of non-normal operators. The problem "when is a hyponormal operator normal?" is studied by several researchers. It is known that a compact hyponormal operator is normal (see [1, 5] for more details). We refer [14] for several other such sufficient conditions. One more important result in this direction is that a hyponormal operator whose spectrum has zero area measure is normal, which follows by Putnam's inequality [17, Theorem 1].

In this article, we consider similar questions by weakening the above mentioned conditions. More precisely we replace the compact operator by an operator in the bigger class, namely $\mathcal{AN}$-class of operators. Another important result we prove is that a hyponormal $\mathcal{AN}$-operator whose essential spectrum and the Weyl spectrum coincide must be normal. To prove these results we first establish a representation of hyponormal $\mathcal{AN}$-operators.

Let H be a complex Hilbert space and T be a bounded linear operator on H. Then T is called *norm attaining* if there exist $x \in H$, $\|x\| = 1$ such that $\|Tx\| = \|T\|$, and *absolutely norm attaining* or $\mathcal{AN}$-operator, if for any closed subspace M of H, the operator $T|_M : M \to H$ is norm attaining, that is there exist $x \in M$, $\|x\| = 1$ such that $\|T|_M x\| = \|Tx\| = \|T|_M\|$. The set of absolutely norm attaining operators is a subclass of norm attaining operators and contains the space of all compact operators, isometries. In general, it contains partial isometries with finite-dimensional null space. This class was first studied by Carvajal and Neves in [6]. Structure of positive absolutely norm attaining operators has been studied in [16, 18, 20]. In [20], a characterization of normal and self-adjoint $\mathcal{AN}$-operators are studied. In

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general, the adjoint of an $\mathcal{AN}$-operator need not be $\mathcal{AN}$ (see [6] for more details), but when the operator is normal, this is indeed true.

In this article, first we study a characterization of normal $\mathcal{AN}$-operators in terms of the essential spectrum. Later we prove a structure theorem for hyponormal $\mathcal{AN}$-operators and as a consequence, we deduce that a compact hyponormal operator is normal. In the end, we prove that if an $\mathcal{AN}$-operator and its adjoint are paranormal then the operator must be normal.

In the remaining part of this section, we introduce basic notions and notations used in the article. In the second section, we give a characterization of normal $\mathcal{AN}$-operators. In the last section, we prove a representation theorem for the hyponormal $\mathcal{AN}$-operators and deduce important consequences.

1.1. **Preliminaries.** We denote the space of all bounded linear operators from H_1 to H_2 by $\mathcal{B}(H_1, H_2)$. For $T \in \mathcal{B}(H_1, H_2)$, $R(T)$ and $N(T)$ denote the range and null space of T, respectively. If $R(T)$ is finite dimensional, then T is called a *finite-rank operator*. A bounded linear operator T is said to be *compact*, if T maps every bounded set in H_1 into a pre-compact set in H_2. The set of all finite-rank operators and compact operators in $\mathcal{B}(H_1, H_2)$ are denoted by $\mathcal{F}(H_1, H_2)$ and $\mathcal{K}(H_1, H_2)$, respectively. In particular $\mathcal{F}(H) := \mathcal{F}(H, H)$ and $\mathcal{K}(H) := \mathcal{K}(H, H)$.

If $T \in \mathcal{B}(H_1, H_2)$, then the *adjoint* operator $T^* : H_2 \to H_1$ is a bounded linear operator satisfying $\langle Tx, y\rangle = \langle x, T^*y\rangle$, $\forall\, x \in H_1$, $y \in H_2$.

For $T \in \mathcal{B}(H)$, $\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible in } \mathcal{B}(H)\}$ is called the *resolvent* set of T and $\sigma(T) = \mathbb{C}\backslash\rho(T)$ is called the *spectrum* of T.

An operator $T \in \mathcal{B}(H)$ is said to be *Fredholm*, if $R(T)$ is closed, $N(T)$ and $N(T^*)$ are finite dimensional. In this case, the index of T is defined by

$$ind(T) = \dim N(T) - \dim N(T^*).$$

The *essential spectrum*, and the *Weyl spectrum* of T are defined by

$$\begin{aligned}\sigma_{ess}(T) :=& \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\},\\ \omega(T) :=& \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm of index } 0\},\end{aligned}$$

respectively. We define

$$\pi_{00}(T) = \{\lambda \in \sigma(T) : \lambda \text{ is an isolated eigenvalue with finite multiplicity}\}.$$

The quantity $m_e(T) := \inf\{\lambda : \lambda \in \sigma_{ess}(|T|)\}$ is called the *essential minimum modulus* of T. For more details about Fredholm theory and the essential spectrum, we refer to [4, 15, 19].

If M is a closed subspace of H, then the unit sphere in M is denoted by $S_M = \{x \in M : \|x\| = 1\}$. For $T \in \mathcal{B}(H)$, M is said to be invariant under T if $TM \subseteq M$. For a non-negative real number r, the open disc and circle with centre 0 and radius r are denoted by $D(0, r) := \{z \in \mathbb{C} : |z| < r\}$ and $C(0, r) := \{z \in \mathbb{C} : |z| = r\}$, respectively.

For $T \in \mathcal{B}(H)$, $m(T) := \inf\{\|Tx\|; x \in H_1,\ \|x\| = 1\}$ is called the *minimum modulus* of T. For more details, we refer to [7, 9].

Let $T \in \mathcal{B}(H)$. Then T is said to be *normal* if $T^*T = TT^*$, *self-adjoint* if $T^* = T$ and *positive* if $\langle Tx, x\rangle \geqslant 0$, $\forall x \in H$.

Definition 1.1. Let $T \in \mathcal{B}(H)$. Then T is called

(1) *hyponormal*, if $TT^* \leqslant T^*T$.
(2) *paranormal*, if $\|Tx\|^2 \leqslant \|T^2x\|\|x\|$, $\forall\, x \in H$.

Note that every hyponormal operator is paranormal.

Theorem 1.2. [1, 5] *Every compact hyponormal operator $T \in \mathcal{B}(H)$ is normal.*

The above result is true for paranormal operators as well, see [12, Theorem 2] for more details.

Lemma 1.3. [13, Theorem 3] *Suppose $T \in \mathcal{B}(H)$ is a paranormal norm attaining operator. Then $N(|T| - \|T\|I)$ is an invariant subspace for T.*

An operator $T \in \mathcal{B}(H)$ is called *absolutely norm attaining* if for every closed subspace $M \subseteq H$, $T|_M$ is norm attaining. We denote the set of all absolutely norm attaining operators in $\mathcal{B}(H_1, H_2)$ by $\mathcal{AN}(H_1, H_2)$ and $\mathcal{AN}(H) := \mathcal{AN}(H, H)$.

First we recall a few important results related to positive $\mathcal{AN}$-operators which we need to prove our results. For positive absolutely norm attaining operators, we have the following characterizations.

Theorem 1.4. [16, Theorem 5.1] *Let H be a complex Hilbert space of arbitrary dimension and let P be a positive operator on H. Then P is an $\mathcal{AN}$-operator if and only if P is of the form $P = \alpha I + K + F$, where $\alpha \geqslant 0$, K is a positive compact operator and F is a self-adjoint finite rank operator.*

We remark that the representation in Theorem 1.4 can be made unique and the value of α can be found out to be $m_e(T)$, the essential minimum modulus of T. For full details, we refer to [20].

In [18] the following representation for normal $\mathcal{AN}$-operators is proved.

Theorem 1.5. [18, Theorem 3.9, Theorem 3.13] *Let $T \in \mathcal{AN}(H)$ be a normal operator. Then there exist $(H_\beta, U_\beta)_{\beta\in\sigma(|T|)}$ such that*

(1) H_β is a reducing subspace for T,
(2) $U_\beta \in \mathcal{B}(H_\beta)$ is a unitary operator.

such that

(1) $H = \bigoplus\limits_{\beta\in\sigma(|T|)} H_\beta$,
(2) $T = \bigoplus\limits_{\beta\in\sigma(|T|)} \beta U_\beta$,
(3) $\sigma(T) \subseteq \bigcup\limits_{\beta\in\sigma(|T|)} \beta\mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

In the above theorem, precisely $H_\beta = N(|T| - \beta I)$. The above theorem implies that every normal $\mathcal{AN}$-operator is direct sum of scalar multiple of unitary operators.

Theorem 1.6. [18, Theorem 2.4] *Let H be an infinite dimensional Hilbert space and $T \in \mathcal{B}(H)$ be positive. Then $T \in \mathcal{AN}(H)$ if and only if $\sigma_{ess}(T)$ is singleton set and $[m(T), m_e(T))$ contains at most finitely many points of $\sigma(T)$.*

2. Normal $\mathcal{AN}$-operators

Here we give a spectral characterization of normal absolutely norm attaining operators in terms of the essential spectrum.

Theorem 2.1. *Let H be an infinite-dimensional Hilbert space and $T \in \mathcal{B}(H)$ be a normal operator. Then the following are equivalent.*

(1) $T \in \mathcal{AN}(H)$.
(2) There exists a non-negative real number $\alpha \geqslant 0$ such that $\sigma_{ess}(T) \subseteq C(0,\alpha)$ and $D(0,\alpha)$ contains at most finitely many points of $\sigma(T)$.

Proof. (1)⇒(2) Let $T \in \mathcal{AN}(H)$. By Theorem 1.5, T can be represented as

$$T = \bigoplus_{\beta \in \sigma(|T|)} \beta U_\beta,$$

where U_β is unitary operator on $H_\beta = N(|T| - \beta I)$. Since $T \in \mathcal{AN}(H)$, we get $|T| \in \mathcal{AN}(H)$ by [16, Lemma 6.2] and consequently $\sigma_{ess}(|T|)$ is singleton set by Theorem 1.6. Thus H_β is finite-dimensional subspace of H for all $\beta \neq m_e(|T|)$ (that is $\beta \neq m_e(T)$). Let us write $\alpha = m_e(T)$. As U_β is a finite rank operator for every $\beta \neq \alpha$, it follows that $\sigma_{ess}(T) \subseteq C(0,\alpha)$.

Also, $\sigma(T) \subseteq \bigcup_{\beta \in \sigma(|T|)} \sigma(\beta U_\beta)$. As $\sigma(U_\beta)$ is finite for $\beta \neq \alpha$ and by Theorem 1.6 there are at most finitely many $\beta \in \sigma(|T|)$ such that $\beta < \alpha$, say $\beta_1, \beta_2, \ldots \beta_m$ (see [18, Theorem 3.9] for details), which imply

$$D(0,\alpha) \cap \sigma(T) = \bigcup_{i=1}^{m} [\sigma(\beta_i U_{\beta_i}) \cap \sigma(T)] \subseteq \bigcup_{i=1}^{m} \beta_i \sigma(U_{\beta_i}).$$

For $1 \leqslant i \leqslant m$, U_{β_i} is a finite-rank operator, which implies $\bigcup_{i=1}^{m} \beta_i \sigma(U_{\beta_i})$ is finite and consequently $D(0,\alpha) \cap \sigma(T)$ is a finite subset of $\mathbb{C}$.

(2)⇒(1) Let $D(0,\alpha) \cap \sigma(T) = \{\gamma_1, \gamma_2, \ldots \gamma_{n_0}\}$ for $n_0 \in \mathbb{N} \cup \{0\}$ and $|\gamma_i| = r_i$ for $1 \leqslant i \leqslant n_0$. By the continuous functional calculus, we know that $\lambda \in \sigma(T)$ implies $|\lambda| \in \sigma(|T|)$. Therefore $r_1, r_2, \ldots r_{n_0} \in \sigma(|T|)$. Also note that by definition $m_e(T) = \alpha$.

We claim that $\sigma(|T|) \cap [0,\alpha) = \{r_1, r_2, \ldots r_{n_0}\}$. If there exist some $r < \alpha$ such that $r \in \sigma(|T|)$, then again by the functional calculus $C(0,r) \cap \sigma(T) \neq \varnothing$. So, r should be one of these r_i, $1 \leqslant i \leqslant n_0$.

Finally, we have that $\sigma_{ess}(|T|) = \{\alpha\}$ and $[0,\alpha) \cap \sigma(|T|)$ is a finite set, namely $\{r_1, r_2, \ldots r_{n_0}\}$. By Theorem 1.6, $|T| \in \mathcal{AN}(H)$ and consequently $T \in \mathcal{AN}(H)$ by [16, Lemma 6.2]. □

Remark 2.2. For non-normal $\mathcal{AN}$ operators part *(2)* of Theorem 2.1 need not hold. The following example illustrates this.

Example 2.3. Consider the right shift operator $R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by $R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$, $\forall (x_n) \in \ell^2(\mathbb{N})$. As R is an isometry, it is an $\mathcal{AN}$-operator. We know that $\sigma(R) = \overline{D(0,1)}$, $\sigma_{ess}(R) = C(0,1)$ and $m_e(R) = 1$. It is easy to observe that $D(0,1)$ contains uncountably many points of $\sigma(R)$. Though $R \in \mathcal{AN}(\ell^2(\mathbb{N}))$, it does not satisfy part *(2)* of Theorem 2.1.

By [20, Proposition 3.4], we know that if $T = T^* \in \mathcal{AN}(H)$, then the essential spectrum of T contains at most two points. Here, in fact by Theorem 2.1 we can characterize self-adjoint $\mathcal{AN}$-operators. A different characterization of self-adjoint $\mathcal{AN}$-operators is given in Theorem 3.3 and Corollary 4.5 of [20].

Corollary 2.4. *Let $T \in \mathcal{B}(H)$ be a self-adjoint operator. Then $T \in \mathcal{AN}(H)$ if and only if there exists $\alpha \geqslant 0$ such that $\sigma_{ess}(T) \subseteq \{-\alpha, \alpha\}$ and $(-\alpha, \alpha) \cap \sigma(T)$ is at most finite.*

Proof. From Theorem 2.1, $T \in \mathcal{AN}(H)$ if and only if there exists $\alpha \geqslant 0$ such that $\sigma_{ess}(T) \subseteq \partial D(0,\alpha))$ and $D(0,\alpha) \cap \sigma(T)$ is at most finite. Since $T = T^*$, we have $\sigma(T) \subseteq [-\|T\|, \|T\|]$. In this case $D(0,\alpha) \cap \sigma(T) = (-\alpha, \alpha) \cap \sigma(T)$ is at most finite and $\sigma_{ess}(T) \subseteq \{-\alpha, \alpha\}$. □

We conclude this section with a comment on absolutely minimum attaining operators. Recall that $T \in \mathcal{B}(H_1, H_2)$ is called *minimum attaining*, if there exists $x \in H_1$ with $\|x\| = 1$ such that $\|Tx\| = m(T)$. An operator T is said to be *absolutely minimum attaining* (or $\mathcal{AM}$-operator, in short) if for every non zero closed subspace M of H, the operator $T|_M : M \to H$ is minimum attaining. We refer [7, 9, 10] and references therein for more detail about $\mathcal{AM}$-operators. The structure of multiplication $\mathcal{AM}$-operators as well as normal $\mathcal{AM}$-operators are discussed in [3].

By imitating the proof of Theorem 2.1 and using [3, Theorem 4.4], we can get the following characterization for normal $\mathcal{AM}$-operators.

Theorem 2.5. *Let $T \in \mathcal{B}(H)$ be a normal operator. The following are equivalent.*

(1) $T \in \mathcal{AM}(H)$.
(2) There exists a real number $\beta \geqslant 0$ such that $Ann(0; \beta, \|T\|)$ contains at most finitely many points of $\sigma(T)$ and $\sigma_{ess}(T) \subseteq C(0, \beta)$, where $Ann(0; \beta, \|T\|) := \{z \in \mathbb{C} : \beta < |z| \leqslant \|T\|\}$.

Remark 2.6. The spectral decomposition of positive $\mathcal{AM}$-operators is established in [3].

3. Hyponormal $\mathcal{AN}$-operators

In this section, we prove a structure theorem for hyponormal $\mathcal{AN}$-operators and deduce a few important consequences.

Lemma 3.1. *Let $T \in \mathcal{B}(H)$ be a non-zero norm attaining hyponormal operator. If $N(|T| - \|T\|I)$ is finite dimensional, then it is a reducing subspace for T. Moreover $T|_{N(|T|-\|T\|I)} = \|T\|U$, where $U \in \mathcal{B}(N(|T| - \|T\|I))$ is a unitary operator.*

Proof. Let $M := N(|T| - \|T\|I)$. By Lemma 1.3, we know that M is an invariant subspace for T and $\frac{T}{\|T\|}$ is an isometry on M. But as M is finite dimensional, we have that $\frac{T}{\|T\|}$ is unitary on M, denote it by U. Hence $T|_M = \|T\|U$.

Since M is an invariant subspace for T, T has the following matrix representation;

$$T = \begin{bmatrix} T|_M & A \\ 0 & B \end{bmatrix},$$

where $A \in \mathcal{B}(M^\perp, M)$, $B \in \mathcal{B}(M^\perp)$. Since T is hyponormal, we have

$$0 \leqslant T^*T - TT^* = \begin{bmatrix} -AA^* & (T|_M)^*A - AB^* \\ A^*T|_M - BA^* & A^*A + B^*B - BB^* \end{bmatrix}.$$

By the positivity of operator matrix, we get $-AA^* \geqslant 0$, which imply $A = 0$. Hence

$$T = \begin{bmatrix} T|_M & 0 \\ 0 & B \end{bmatrix}. \tag{3.1}$$

That is, M is a reducing subspace for T. □

Remark 3.2. Note that B is hyponormal in the above representation (3.1). In fact, T is normal if and only if B is normal.

Proposition 3.3. *Let $T \in \mathcal{B}(H)$ be a non-zero hyponormal $\mathcal{AN}$-operator with $\sigma_{ess}(|T|) = \{\|T\|\}$. If $\pi_{00}(|T|)$ is non-empty and $\pi_{00}(|T|) = \{\lambda_i\}_{i=1}^{m_0}$ for some $m_0 \in \mathbb{N}$, then*

(1) $H = H_1 \oplus H_2$, where $H_1 = N(|T| - \|T\|I)$, $H_2 = \bigoplus_{i=1}^{m_0} N(|T| - \lambda_i I)$.

(2) T has the following representation with respect to $H_1 \oplus H_2$.

$$T = \begin{array}{c} \begin{matrix} H_1 & H_2 \end{matrix} \\ \begin{pmatrix} \|T\|S_0 & A \\ 0 & B \end{pmatrix} \begin{matrix} H_1 \\ H_2 \end{matrix} \end{array}$$

where

(a) $S_0 \in \mathcal{B}(H_1)$ is an isomtery.

(b) A, B are finite-rank operators with $S_0^*A = 0$ and $(A+B)^*(A+B) = \bigoplus_{i=1}^{m_0} \lambda_i^2 I_{N(|T|-\lambda_i I)}$.

If $\pi_{00}(T)$ is an empty set then $T = \|T\|S_0$.

Proof. Without loss of generality, we assume that $T = S|T|$, where $S \in \mathcal{B}(H)$ is an isometry. To see this, let $T = V|T|$ be the polar decomposition of T, where V is a partial isometry with the initial space $N(T)^\perp$ and the final

space $\overline{R(T)}$. As T is hyponormal we have that $N(T) \subseteq N(T^*) = R(T)^\perp$, so there exist a partial isometry W with the initial space $N(T)$ and the final space $R(T)^\perp$. Then $S := V + W$ is an isometry on H and $T = S|T|$.

Note that as $|T| \in \mathcal{AN}(H)$, we have $\|T\| \in \sigma_p(|T|)$. Since $\sigma_{ess}(|T|) = \{\|T\|\}$, it is clear that $N(|T| - \|T\|I)$ is infinite dimensional subspace of H. By [18, Lemma 3.4], $N(|T| - \|T\|I) = N(T^*T - \|T\|^2 I)$ is invariant under T. We observe from the following equation that $N(|T| - \|T\|I)$ is invariant under S;

$$Sx = \frac{S|T|x}{\|T\|} = \frac{Tx}{\|T\|} \in N(|T| - \|T\|I), \ \forall x \in N(|T| - \|T\|I).$$

Since $|T| \in \mathcal{AN}(H)$, it follows that $\pi_{00}(|T|)$ is at most finite by Theorem 1.6. If $\pi_{00}(|T|)$ is empty set, then $H_2 = \{0\}, H = H_1$ and $T = \|T\|S$.

Now, we assume that $\pi_{00}(|T|)$ is non-empty and equal to $\{\lambda_i\}_{i=1}^{m_0}$ for some $m_0 \in \mathbb{N}$, where $\lambda_i < \|T\|$. Thus we have

$$S = \begin{matrix} \\ H_1 \\ H_2 \end{matrix} \begin{matrix} \begin{matrix} H_1 & H_2 \end{matrix} \\ \begin{pmatrix} S|_{H_1} & S_1 \\ 0 & S_2 \end{pmatrix} \end{matrix}. \tag{3.2}$$

Note that $S_1 = P_{H_1} S|_{H_2}$ and $S_2 = P_{H_2} S|_{H_2}$. Also

$$|T| = \begin{bmatrix} \|T\| I_{H_1} & 0 \\ 0 & F \end{bmatrix},$$

where $F = \bigoplus_{i=1}^{m_0} \lambda_i I_{M_i}$ and $M_i = N(|T| - \lambda_i I)$ is finite dimensional subspace of H. From this we conclude that

$$T = \begin{bmatrix} S|_{H_1} & S_1 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} \|T\| I_{H_1} & 0 \\ 0 & F \end{bmatrix} = \begin{bmatrix} \|T\| S|_{H_1} & S_1 F \\ 0 & S_2 F \end{bmatrix}.$$

Now we write $A := S_1 F$, $B := S_2 F$ and $S_0 = S|_{H_1}$. As T is hyponormal operator, we have

$$\begin{aligned} 0 &\leqslant T^*T - TT^* \\ &= \begin{bmatrix} \|T\|^2 I_{H_1} - \|T\|^2 P_{R(S_0)} - AA^* & \|T\| S_0^* A - AB^* \\ \|T\| A^* S_0 - BA^* & A^*A + B^*B - BB^* \end{bmatrix}. \end{aligned}$$

Thus $AA^* \leqslant \|T\|^2 P_{R(S_0)^\perp}$, which gives $\overline{R(A)} = \overline{R(AA^*)} \subseteq R(S_0)^\perp = N(S_0^*)$. Consequenlty $S_0^* A = 0$. We have $A+B = (S_1+S_2)F$, thus $(A+B)^*(A+B) = F^2 = \bigoplus_{i=1}^{m_0} \lambda_i^2 I_{M_i}$, as $S_1^* S_1 + S_2^* S_2 = I_{H_2}$ by equation (3.2). □

Corollary 3.4. *The operator T in Proposition 3.3 is normal if and only if S_0 is unitary.*

Proof. If S_0 is unitary then by the condition $S_0^* A = 0$, we have $A = 0$. In this case, B is a hyponormal operator. As it is finite-rank operator, it must be normal by Theorem 1.2. Hence T is normal.

Conversely assume that T is a normal operator. Then $N(|T| - \|T\|I) = N(T^*T - \|T\|^2 I) = N(TT^* - \|T\|^2 I)$ is a reducing subspace for T, that is H_1 is a reducing subspace for T. Let $T = S|T|$, where S is as in Proposition 3.3. Since T is normal, we can easily prove that $S|T| = |T|S$. From this and the fact that H_1 reduces T, we can conclude that H_1 is reducing for S also. Hence we have

$$S = \begin{bmatrix} S_0 & 0 \\ 0 & S_2 \end{bmatrix}.$$

As a result $A = 0 \cdot F = 0$ and the normality of T forces that S_0 to be unitary. □

Theorem 3.5. *Let $T \in \mathcal{AN}(H)$ be a hyponormal operator. Then*

(1) there exists closed subspaces H_0, H_1, H_2 with $\dim(H_2) < \infty$ such that $H = H_0 \oplus H_1 \oplus H_2$.

(2) T has the following representation with respect to $H_0 \oplus H_1 \oplus H_2$;

$$T = \begin{array}{c} \begin{matrix} H_0 & H_1 & H_2 \end{matrix} \\ \begin{pmatrix} S_0 & 0 & 0 \\ 0 & \lambda S_1 & A \\ 0 & 0 & S_2 \end{pmatrix} \end{array} \begin{matrix} H_0 \\ H_1 \\ H_2 \end{matrix},$$

where $\sigma_{ess}(|T|) = \{\lambda\}$, for some $\lambda \in \mathbb{R}$ and

(a) $S_0 = \bigoplus_{i=1}^{n_0} \lambda_i U_i$ for some unitary operator $U_i \in \mathcal{B}(N(|T| - \lambda_i I))$, if $(\lambda, \|T\|] \cap \sigma(|T|) = \{\lambda_i\}_{i=1}^{n_0}$ for some $n_0 \in \mathbb{N} \cup \{\infty\}$ or $S_0 = 0$ if $(\lambda, \|T\|] \cap \sigma(|T|)$ is empty set.

(b) $S_1 \in \mathcal{B}(H_1)$ is an isometry.

(c) A, S_2 are finite-rank operators with $S_1^* A = 0$ and

$$(A + S_2)^*(A + S_2) = \bigoplus_{j=1}^{m_0} \delta_j^2 I_{N(|T| - \delta_j I)},$$

if $[0, \lambda) \cap \sigma(|T|) = \{\delta_j : 1 \leqslant j \leqslant m_0\}$ for some $m_0 \in \mathbb{N}$.

Proof. Let $T \in \mathcal{AN}(H)$. Then $|T| \in \mathcal{AN}(H)$ and hence $\sigma_{ess}(|T|)$ is singleton, say $\{\lambda\}$ and $\sigma(|T|)$ is countable, by Theorem 1.6. Again using Theorem 1.6, we know that $[0, \lambda)$ contains at most finitely many points of $\sigma(|T|)$ and $(\lambda, \|T\|]$ contains at most countably many points of $\sigma(|T|)$.

First, we assume that $\lambda < \|T\|$ and consequently $(\lambda, \|T\|] \cap \sigma(|T|)$ is non-empty and $(\lambda, \|T\|] \cap \sigma(|T|) = \{\lambda_i\}_{i=1}^{n_0}$ for some $n_0 \in \mathbb{N} \cup \{\infty\}$. Now, define $H_0 = \bigoplus_{i=1}^{n_0} N(|T| - \lambda_i I)$ and $H_1 = N(|T| - \lambda I)$.

Without loss of generality, we assume that $\lambda_1 = \|T\|$. By Lemma 3.1, $N(|T| - \|T\|I)$ is reducing subspace for T and $T|_{N(|T| - \|T\|I)} = \|T\|U$ for some unitary operator $U \in \mathcal{B}(N(|T| - \|T\|I))$. Again $T|_{(N(|T| - \|T\|I))^\perp}$ is hyponormal $\mathcal{AN}$-operator. Now repeating the same procedure, we get $S_0 := T|_{H_0} = \bigoplus_{i=1}^{n_0} \lambda_i U_i$, where $U_i \in \mathcal{B}(N(|T| - \lambda_i I))$ is unitary operator for every $1 \leqslant i \leqslant n_0$.

As H_0 is a reducing subspace for T, we get $T|_{H_0^\perp}$ is a hyponormal $\mathcal{AN}$-operator and $\|T|_{H_0^\perp}\| \leqslant \lambda$. Now, it is enough to look at the structure of $T|_{H_0^\perp}$.

We consider the following four cases which exhaust all possibilities.

Case (1): λ is an eigenvalue of $|T|$ with infinite multiplicity but not a limit point of $\sigma(|T|)$:
In this case $(\lambda, \|T\|] \cap \sigma(|T|)$ is a finite set by [16, Theorem 3.8] and consequently $n_0 \in \mathbb{N}$. In this case $\|T|_{H_0^\perp}\| = |\lambda|$ and $\{\lambda\} = \sigma_{ess}(T|_{H_0^\perp})$. Using Proposition 3.3, we get $H_0^\perp = H_1 \oplus H_2$, where H_2 is a finite dimensional space and $H_1 = N(|T| - \lambda I)$. Hence we have

$$T|_{H_0^\perp} = \begin{bmatrix} \lambda S_1 & A \\ 0 & S_2 \end{bmatrix},$$

where S_1, A and S_2 satisfy conditions (b) and (c). Therefore T has the following representation satisfying (a), (b), (c).

$$T = \begin{array}{c} \begin{matrix} H_0 & H_1 & H_2 \end{matrix} \\ \begin{pmatrix} S_0 & 0 & 0 \\ 0 & \lambda S_1 & A \\ 0 & 0 & S_2 \end{pmatrix} \begin{matrix} H_0 \\ H_1 \\ H_2 \end{matrix} \end{array}.$$

Case (2): λ is not an eigenvalue of $|T|$ but it is a limit point of $\sigma(|T|)$:
In this case, $(\lambda, \|T\|] \cap \sigma(|T|) = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ is an infinite set by [16, Theorem 3.8], and $H_1 = \{0\}$. As λ is not an eigenvalue of T, we get $\|T|_{H_0^\perp}\| < \lambda$. If $\sigma(|T|) \cap [0, \lambda)$ is empty, then $H_2 = \{0\}$ and $T = S_0$.

If $[0, \lambda) \cap \sigma(|T|)$ is non-empty and is equal to $\{\delta_j\}_{j=1}^{m_0}$ for some $m_0 \in \mathbb{N}$, then take $H_2 = \overset{m_0}{\underset{i=1}{\oplus}} N(|T| - \delta_i I)$ and

$$T = \begin{bmatrix} S_0 & 0 \\ 0 & S_2 \end{bmatrix},$$

where $S_2 = \overset{m_0}{\underset{j=1}{\oplus}} \delta_j V_j$ and $V_j \in \mathcal{B}\left(N(|T| - \delta_j I)\right)$ is a unitary operator.

Case (3): λ is neither an eigenvalue of $|T|$ nor a limit point of $\sigma(|T|)$:
Then $(\lambda, \|T\|] \cap \sigma(|T|)$ is a finite set by [16, Theorem 3.8], thus $n_0 \in \mathbb{N}$ and $H_1 = \{0\}$.

If $[0, \lambda) \cap \sigma(|T|)$ is empty set, then $H = H_0$ and $T = S_0$.

If $[0, \lambda) \cap \sigma(|T|)$ is non-empty and equal to $\{\delta_1, \delta_2, \ldots \delta_{m_0}\}$ for some $m_0 \in \mathbb{N}$, then $H = H_0 \oplus H_2$ is finite dimensional Hilbert space, where $H_2 = \overset{m_0}{\underset{i=1}{\oplus}} N(|T| - \delta_i I)$. As H_0 is a reducing subspace for T, thus we have

$$T = \begin{bmatrix} T|_{H_0} & 0 \\ 0 & T|_{H_0^\perp} \end{bmatrix}.$$

Both $T|_{H_0}$ and $T|_{H_0^\perp}$ are hyponormal finite rank operators, hence normal. By Theorem 1.5

$$T|_{H_0} = \bigoplus_{i=1}^{n_0} \lambda_i U_i \text{ and } T|_{H_0^\perp} = \bigoplus_{j=1}^{m_0} \delta_j V_j,$$

where $U_i \in \mathcal{B}(N(|T| - \lambda_i I))$ and $V_j \in \mathcal{B}(N(|T| - \delta_j I))$ are unitary operators.

Case (4)) λ is an eigenvalue of $|T|$ also it is a limit point of $\sigma(|T|)$: Here $(\lambda, \|T\|] \cap \sigma(|T|)$ is an infinite set, say $\{\lambda_1, \lambda_2, \ldots\}$ by [16, Theorem 3.8]. In this case $\|T|_{H_0^\perp}\| = \lambda$.

If $[0, \lambda) \cap \sigma(|T|)$ is non-empty and equal to $\{\delta_1, \delta_2, \ldots \delta_{m_0}\}$ for some $m_0 \in \mathbb{N}$, then take $H_2 := \bigoplus_{i=1}^{m_0} N(|T| - \delta_i I)$ and we have

$$T = \begin{bmatrix} S_0 & 0 & 0 \\ 0 & \lambda S_1 & A \\ 0 & 0 & S_2 \end{bmatrix}.$$

Using Lemma 3.1 and Proposition 3.3, we get either S_1 is unitary or isometry depending on the multiplicity of the eigenvalue λ. Note that $A = 0$ if λ is of finite multiplicity but it need not be the zero operator, if λ is of infinite multiplicity.

If $[0, \lambda) \cap \sigma(|T|)$ is empty set, then $H_2 = \{0\}$ and

$$T = \begin{bmatrix} S_0 & 0 \\ 0 & \lambda S_1 \end{bmatrix}.$$

If $\lambda = \|T\|$, then the result follows from Proposition 3.3 □

Corollary 3.6. *Let $T \in \mathcal{B}(H)$ satisfy the assumptions of Theorem 3.5. Then T is normal if and only if S_1 is a unitary operator.*

Proof. Let S_1 be a unitary operator. As $S_1^* A = 0$ and S_1 is unitary, we get $A = 0$. In this case, S_2 is a hyponormal operator. As it is finite-rank operator, it must be normal by Theorem 1.2. Hence T is normal.

Conversely, suppose that T is a normal operator. As H_1 is a reducing subspace for T, this implies $T|_{H_1^\perp}$ is normal and satisfy the assumptions of Corollary 3.4. Hence S_1 is a unitary operator and $A = 0$. □

Remark 3.7. Let $T \in \mathcal{AN}(H)$ be a hyponormal operator. From Theorem 3.5, we observe that $\sigma(T) \subseteq \overline{D(0, \lambda) \cup \left(\bigcup_{i=1}^{n_0} C(0, \lambda_i) \right)}$, where $\sigma_{ess}(|T|) = \{\lambda\}$ and $(\lambda, \|T\|] \cap \sigma(|T|) = \{\lambda_1, \lambda_2, \ldots \lambda_{n_0}\}$, for some $n_0 \in \mathbb{N} \cup \{\infty\}$.

Now, we demonstrate Theorem 3.5 with an example.

Example 3.8. Define $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$T(x_1, x_2, \ldots) = \left(\frac{x_1}{2}, 0, \frac{x_2}{2}, x_3, x_4, \ldots \right), \ \forall\, (x_1, x_2, \ldots) \in \ell^2(\mathbb{N}).$$

Then

$$T^*(x_1, x_2, \ldots) = \left(\frac{x_1}{2}, \frac{x_3}{2}, x_4, \ldots\right), \forall\, (x_1, x_2, \ldots) \in \ell^2(\mathbb{N}).$$

Clearly $\|T^*(x_n)\| \leqslant \|T(x_n)\|, \forall\, (x_n) \in \ell^2(\mathbb{N})$, hence T is hyponormal. By simple computation, we get

$$T^*T(x_1, x_2, \ldots) = \left(\frac{x_1}{4}, \frac{x_2}{4}, x_3, \ldots\right).$$

It is easy to see that $T^*T = I - F$, where

$$F(x_n) = \left(\frac{3x_1}{4}, \frac{3x_2}{4}, 0 \ldots\right), \forall\, (x_n) \in \ell^2(\mathbb{N}).$$

By Theorem 1.4, we conclude that T^*T is $\mathcal{AN}$ and hence T is $\mathcal{AN}$ by [20, Corollary 2.11]. Now choose $H_1 := \overline{\text{span}}\{e_3, e_4, \ldots\}$ and $H_2 = \text{span}\{e_1, e_2\}$, then T can be represented as

$$T = \begin{array}{c} \begin{matrix} H_1 & \quad H_2 \end{matrix} \\ \left(\begin{array}{c|cc} R & \multicolumn{2}{c}{A} \\ \hline 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array} \begin{matrix} H_1 \\ \\ H_2 \end{matrix}$$

where $R(x_3, x_4, \ldots) = (0, x_3, x_4, \ldots)$ and $A(x_1, x_2) = (x_2/2, 0, \ldots)$.

The following examples illustrates the fact that the Theorem 3.5 can be improved in terms of the unitary operators for some particular cases but not in general.

Example 3.9. Define $T_1, T_2 : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$\begin{aligned} T_1(x_1, x_2, x_3, x_4, \ldots) &= (0, x_1, x_2, 2x_3, 2x_4, \ldots), \forall\, (x_n) \in \ell^2(\mathbb{N}) \\ T_2(x_1, x_2, x_3, x_4, \ldots) &= (x_1, 2x_2, 0, 3x_3, 3x_4, \ldots), \forall\, (x_n) \in \ell^2(\mathbb{N}). \end{aligned}$$

Let $\{e_n\}_{n\in\mathbb{N}}$ be the standard orthonormal basis for $\ell^2(\mathbb{N})$. For $H_1 := \overline{\text{span}}\{e_3, e_4, \ldots\}$ and $H_2 = \text{ span } \{e_1, e_2\}$, we have

$$T_1 = \begin{array}{c} \begin{matrix} H_1 & H_2 \end{matrix} \\ \begin{pmatrix} 2R_1 & A_1 \\ 0 & B_1 \end{pmatrix} \end{array} \begin{matrix} H_1 \\ H_2 \end{matrix} \text{ and } T_2 = \begin{array}{c} \begin{matrix} H_1 & H_2 \end{matrix} \\ \begin{pmatrix} 3R_1 & 0 \\ 0 & B_2 \end{pmatrix} \end{array} \begin{matrix} H_1 \\ H_2 \end{matrix}$$

where

$$\begin{aligned} R_1(x_3, x_4, \ldots) &= (0, x_3, x_4, \ldots), \forall\, (x_3, x_4, \ldots) \in H_1, \\ A_1(x_1, x_2) &= (x_2, 0, 0, \ldots), \forall\, (x_1, x_2) \in H_2, \\ B_1(x_1, x_2) &= (0, x_1), \forall\, (x_1, x_2) \in H_2, \\ B_2(x_1, x_2) &= (x_1, 2x_2), \forall\, (x_1, x_2) \in H_2. \end{aligned}$$

Note that $B_2 = I_{\{x_1\}} \oplus 2I_{\{x_2\}}$. Observe that B_1 is nilpotent and B_2 is the direct sum of scalar multiple of unitary operators.

We deduce the following well known result from Theorem 3.5.

Corollary 3.10. *Let $T \in \mathcal{K}(H)$ be a hyponormal operator. Then T is normal*

Proof. If T is compact, then $\lambda = 0$ in Theorem 3.5. Hence $S_2 = 0$. So $T = S_0 = \overset{n_0}{\underset{i=1}{\oplus}} \lambda_i U_i$ for $n_0 \in \mathbb{N} \cup \{\infty\}$, which is clearly a normal operator. □

Corollary 3.11. *If $T \in \mathcal{AN}(H)$ is a hyponormal operator with $\sigma_{ess}(T) = \omega(T)$, then T must be normal.*

Proof. By Theorem 3.5, we can write T as

$$T = \begin{matrix} \\ H_0 \\ H_1 \\ H_2 \end{matrix} \begin{matrix} \begin{matrix} H_0 & H_1 & H_2 \end{matrix} \\ \begin{pmatrix} S_0 & 0 & 0 \\ 0 & \lambda S_1 & A \\ 0 & 0 & S_2 \end{pmatrix} \end{matrix},$$

where S_0, S_1, S_2, A and λ are as defined in Theorem 3.5. Note that

$$T = \begin{bmatrix} S_0 & 0 & 0 \\ 0 & \lambda S_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_2 \end{bmatrix}.$$

Write

$$\tilde{T} = \begin{bmatrix} S_0 & 0 & 0 \\ 0 & \lambda S_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This implies that $\sigma_{ess}(\tilde{T}) \subseteq \sigma(\tilde{T}) \subseteq \sigma(S_0) \cup \sigma(\lambda S_1) \cup \{0\}$ and consequently $\text{Area}(\sigma_{ess}(\tilde{T})) \leqslant \text{Area}(\sigma(S_0)) + \text{Area}(\sigma(\lambda S_1))$ by [17]. Here $\text{Area}(A)$ is the planar Lebesgue measure of A.

Since A and S_2 are finite-rank operators, we have $\sigma_{ess}(T) = \sigma_{ess}(\tilde{T})$ and $\omega(T) = \omega(\tilde{T})$, so $\sigma_{ess}(\tilde{T}) = \omega(\tilde{T})$. Also note that $\text{ind}(\tilde{T} - \delta I) = \text{ind}(\lambda S_1 - \delta I_{H_1})$ for every $\delta \in \mathbb{C}$. We know that $\sigma_{ess}(\tilde{T}) = \omega(\tilde{T})$ thus $\sigma_{ess}(\lambda S_1) = \omega(\lambda S_1)$. Since S_0 is the direct sum of scalar multiple of unitary operators, we have $\sigma_{ess}(S_0) = \omega(S_0)$. As both S_0 and λS_1 are hyponormal operators, by [8, Theorem 3.1] we have $\sigma(S_0) \backslash \omega(S_0) = \pi_{00}(S_0)$ and $\sigma(\lambda S_1) \backslash \omega(\lambda S_1) = \pi_{00}(\lambda S_1)$.

Since $\sigma_{ess}(S_0)$, $\sigma_{ess}(\lambda S_1) \subseteq C(0, \lambda)$, we have

$$\begin{aligned} \text{Area}(\sigma(S_0)) =& \text{Area}(\omega(S_0)) = \text{Area}(\sigma_{ess}(S_0)) = 0 \text{ and} \\ \text{Area}(\sigma(\lambda S_1)) =& \text{Area}(\omega(\lambda S_1)) = \text{Area}(\sigma_{ess}(\lambda S_1)) = 0. \end{aligned}$$

Again by [8, Theorem 3.1], we know that $\sigma(T) \backslash \omega(T) = \pi_{00}(T)$ and it follows that

$$\begin{aligned} \text{Area}(\sigma(T)) = \text{Area}(\omega(T)) = \text{Area}(\sigma_{ess}(T)) &= \text{Area}(\sigma_{ess}(\tilde{T})) \\ &\leqslant \text{Area}(\sigma(S_0)) + \text{Area}(\sigma(\lambda S_1)) \\ &= 0. \end{aligned}$$

Since T is hyponormal operator, by [17, Theorem 1] we have

$$\|T^*T - TT^*\| \leqslant \frac{1}{\pi}\text{Area}(\sigma(T)) = 0.$$

Hence T is normal. □

In [18, Question 3.12], it is asked that when a paranormal $\mathcal{AN}$-operator is normal? In the same paper, sufficient conditions are given for this problem. Here we prove that we can drop some conditions and still we get normality. It is to be mentioned that in [22, Corollary 3] proved that the paranormality of T and T^* imply that T is normal. Here we give a direct proof by assuming extra condition, namely the $\mathcal{AN}$-property.

Theorem 3.12. *Let T, T^* be paranormal and $T \in \mathcal{AN}(H)$. Then T is normal.*

Proof. If $|T|$ has no eigenvalue with infinite multiplicity, then by [18, Theorem 3.13], T must be normal.

Next assume, that $|T|$ has eigenvalue with infinite multiplicity. By [18, Theorem 3.9], T can be represented as

$$T = \left(\bigoplus_{\substack{\beta \in \sigma(|T|) \\ \beta \neq \beta_0}} \beta U_\beta \right) \oplus \beta_0 U_{\beta_0},$$

where $\sigma_{ess}(|T|) = \{\beta_0\}$. Here U_β is unitary for all $\beta \neq \beta_0$.

Since $|T|$ has an eigenvalue with infinite multiplicity, then it must be β_0. In this case U_{β_0} is an isometry and $H_{\beta_0} = N(|T| - \beta_0 I)$ is a reducing subspace for T. Moreover, $T|_{H_{\beta_0}} = \beta_0 U_{\beta_0}$. To show T is normal, it is enough to prove that U_{β_0} is unitary. As $\beta_0 U_{\beta_0} = T|_{H_{\beta_0}}$ and the restriction of a paranormal operator to any invariant subspace is also paranormal, thus both the operators U_{β_0} and $U^*_{\beta_0}$ are paranormal. By [11, Theorem 1], we conclude that U_{β_0} is unitary and hence T is normal. □

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