

# A Note on Even Cycles and Quasi-Random Tournaments

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## Abstract

A cycle  $C = \{v_1, v_2, \dots, v_1\}$  in a tournament  $T$  is said to be even, if when walking along  $C$ , an even number of edges point in the wrong direction, that is, they are directed from  $v_{i+1}$  to  $v_i$ . In this short paper, we show that for every fixed even integer  $k \geq 4$ , if close to half of the  $k$ -cycles in a tournament  $T$  are even, then  $T$  must be quasi-random. This resolves an open question raised in 1991 by Chung and Graham [5].

## 1 Introduction

Quasi-random (or pseudo-random) objects are *deterministic* objects that possess the properties we expect truly *random* ones to have. One of the most surprising phenomena in this area is the fact that in many cases, if an object satisfies a single *deterministic* property then it must “behave” like a typical random object in many useful aspects. In this paper we will study one such phenomenon related to quasi-random tournaments. The notion of quasi-randomness has been widely studied for different combinatorial objects, like graphs, hypergraphs, groups and set systems [4, 6, 7, 9, 13, 14]. We refrain from giving a detailed discussion of this area in this short paper, and instead refer the reader to the surveys of Gowers [8] and Krivelevich and Sudakov [12] for more details and references.

A directed graph  $D = (V, E)$  consists of a set of vertices and a set of directed edges  $E \subseteq V \times V$ . We use the ordered pair  $(u, v) \in V \times V$  to denote directed edge from  $u$  to  $v$ . A tournament  $T = (V, E)$  is a directed graph such that given any two distinct vertices  $u, v \in V$ , there exists exactly one of the two directed edges  $(u, v)$  or  $(v, u)$  in  $E(T)$ . One can also think of a tournament as an orientation of an underlying complete graph on  $V$ . We shall use  $n$  to denote  $|V|$ .

Consider a tournament  $T = (V, E)$ . For  $Y \subseteq V$ , and  $v \in V$ , let  $d^+(v, Y)$  denote the number of directed edges going from  $v$  to  $Y$  and  $d^-(v, Y)$  denote the number of directed edges going from

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$Y$  to  $v$ . A purely random tournament is one where for each pair of distinct vertices  $u$  and  $v$  of  $V$ , the directed edge between them is chosen randomly to be either  $(u, v)$  or  $(v, u)$  with probability  $1/2$ . It is clear that in a random tournament  $T$ , we have  $\sum_{v \in X} |d^+(v, Y) - d^-(v, Y)| = o(n^2)$  for all  $X, Y \subseteq V(T)$ . Let us define the corresponding property  $\mathcal{Q}$  as follows:

**Definition 1.1.** *A tournament  $T$  on  $n$  vertices satisfies property  $\mathcal{Q}$  if*

$$\sum_{v \in X} |d^+(v, Y) - d^-(v, Y)| = o(n^2) \quad \text{for all } X, Y \subseteq V(T).$$

The notion of quasi-randomness in tournaments was introduced by Chung and Graham [5]. They defined several properties of tournaments, all of which are satisfied by purely random tournaments, including the property  $\mathcal{Q}$  above. They also showed that all these properties are equivalent, namely, if a tournament satisfies one of these properties, then it must also satisfy all the other. They then defined a tournament to be quasi-random if it satisfies any (and therefore, all) of these properties. For the sake of brevity, we will focus on property  $\mathcal{Q}$  (defined above) which will turn out to be the easiest one to work with in the context of the present paper.

Another property studied in [5] was related to even cycles in tournaments. A  $k$ -cycle is an ordered sequence of vertices  $(v_1, v_2, \dots, v_k, v_1)$  such that no vertex is repeated immediately in the sequence. That is,  $v_i \neq v_{i+1}$  for all  $i \leq k - 1$  and  $v_k \neq v_1$ . We say that a  $k$ -cycle (for an integer  $k \geq 2$ ) is even if as we traverse the cycle, we see an even number of directed edges opposite to the direction of the traversal. If a  $k$ -cycle is not even, we call it odd. Let  $E_k(T)$  denote the number of even  $k$ -cycles in a tournament  $T$ . Clearly, the number of  $k$ -cycles in an  $n$ -vertex tournament is  $n^k - o(n^k)$ . In fact, it is not hard to see that the exact number is given by  $(n-1)^k + (-1)^k(n-1)$  (see Section 3). In a random tournament, we expect about half of the  $k$ -cycles to be even. This motivated Chung and Graham [5] to define the following property.

**Definition 1.2.** *A tournament  $T$  on  $n$  vertices satisfies<sup>1</sup> property  $\mathcal{P}(k)$  if  $E_k(T) = (1/2 \pm o(1))n^k$ .*

Notice that when  $k$  is an odd integer,  $E_k(T)$  is *exactly* half the number of  $k$ -cycles in  $T$ , since an even cycle becomes odd upon traversal in the reverse direction. Hence, property  $\mathcal{P}(k)$  cannot be equivalent to property  $\mathcal{Q}$  when  $k$  is odd.

Chung and Graham [5] proved that  $\mathcal{P}(4)$  is quasi-random. In other words, a tournament has (approximately) the correct number of even 4-cycles we expect to find in a random tournament, if and only if it satisfies property  $\mathcal{Q}$ . A question left open in [5] was whether  $\mathcal{P}(k)$  is equivalent to  $\mathcal{Q}$  for all even  $k \geq 4$ . Our main result answers this positively by proving the following.

**Theorem 1.** *The following holds for every fixed even integer  $k \geq 4$ : A tournament satisfies property  $\mathcal{Q}$  if and only if it satisfies property  $\mathcal{P}(k)$ .*

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<sup>1</sup>Observe that our definition of a  $k$ -cycle allows repeated vertices in the cycle. Note however, that forbidding repeated vertices (that is, requiring the  $k$ -cycles to be simple) would have resulted in the same property  $\mathcal{P}(k)$  since the number of  $k$ -cycles with repeated vertices is  $o(n^k)$ . Allowing repeated vertices simplifies some of the notation.

As usual, when we say that property  $\mathcal{Q}$  implies property  $\mathcal{P}(k)$  we mean that for every  $\varepsilon$  there is a  $\delta = \delta(\varepsilon)$ , such that any large enough tournament satisfying  $\sum_{v \in X} |d^+(v, Y) - d^-(v, Y)| \leq \delta n^2$  for all  $X, Y$  has  $(1/2 \pm \varepsilon)n^k$  even cycles. The meaning of  $\mathcal{P}(k)$  implies  $\mathcal{Q}$  is defined similarly.

## 2 Proof of Main Result

To prove Theorem 1, we shall go through a spectral characterization of quasi-randomness. We use the following adjacency matrix  $A$  to represent the tournament  $T$ . For every  $u, v \in V$

$$A_{u,v} = \begin{cases} 1 & \text{if } (u, v) \in E(T) \\ -1 & \text{if } (v, u) \in E(T) \\ 0 & \text{if } u = v \end{cases}$$

A key observation that we will use is that the matrix  $A$  is skew-symmetric. Recall that a real skew symmetric matrix can be diagonalized and all its eigenvalues are purely imaginary. It follows that all the eigenvalues of  $A^2$  are non-positive. This implies the following claim, which will be crucial in our proof.

**Claim 2.1.** *For  $k \equiv 2 \pmod{4}$ , all the eigenvalues of  $A^k$  are non-positive. For  $k \equiv 0 \pmod{4}$ , all the eigenvalues of  $A^k$  are non-negative.*

For a matrix  $M$ , we let  $\text{tr}(M) = \sum_{i=1}^n M_{i,i}$  denote the trace of the matrix  $M$ . Before we prove Lemmas 2.3 and 2.4, we make the following claim.

**Claim 2.2.** *Let  $A$  be the adjacency matrix of the tournament  $T$ . Then for an even integer  $k \geq 4$ , we have*

$$\text{tr}(A^k) = 2\mathbf{E}_k(T) - (n-1)^k - (n-1).$$

*In particular,  $T$  satisfies the property  $\mathcal{P}(k)$  if and only if  $|\text{tr}(A^k)| = o(n^k)$ .*

*Proof.* Notice that the  $(u, u)$ -th entry of  $A^k$  is the number of even  $k$ -cycles starting and ending at  $u$  minus the number of odd  $k$ -cycles starting and ending at  $u$ . So the sum of all diagonal entries,  $\text{tr}(A^k)$ , is the difference between all labeled even  $k$ -cycles and all labeled odd  $k$ -cycles. Recall that the total number of  $k$ -cycles is  $(n-1)^k + (n-1)$  for even  $k$ . Thus we have that  $\text{tr}(A^k) = 2\mathbf{E}_k(T) - (n-1)^k - (n-1)$ .

We have  $\text{tr}(A^k) = 2\mathbf{E}_k(T) - n^k + o(n^k)$ . Notice that  $T$  satisfies property  $\mathcal{P}(k)$  when  $\mathbf{E}_k(T) = (1/2 \pm o(1))n^k$ , which happens if and only if  $|\text{tr}(A^k)| = o(n^k)$ .  $\square$

We are now ready to prove the first direction of Theorem 1.

**Lemma 2.3.** *Let  $k \geq 4$  be an even integer. If a tournament satisfies  $\mathcal{P}(k)$  then it satisfies  $\mathcal{Q}$ .*

*Proof.* Let  $\lambda_1(A), \dots, \lambda_n(A)$  be the eigenvalues of  $A$  sorted by their absolute value, so that  $\lambda_1(A)$  has the largest absolute value. We first claim that  $|\lambda_1(A)| = o(n)$ . Assume first that  $k \equiv 0 \pmod{4}$ . Then by Claim 2.1 all the eigenvalues of  $A^k$  are non-negative, implying that

$$\operatorname{tr}(A^k) = \sum_{i=1}^n \lambda_i(A^k) \geq \lambda_1(A^k) = \lambda_1(A)^k. \quad (1)$$

Now, since we assume that  $T$  satisfies  $\mathcal{P}(k)$ , we get from Claim 2.2 that  $|\operatorname{tr}(A^k)| = o(n^k)$ . Equation (1) now implies that  $|\lambda_1(A)| = o(n)$ . If  $k \equiv 2 \pmod{4}$ , then since Claim 2.1 tells us that all eigenvalues are non-positive, we have

$$\operatorname{tr}(A^k) = \sum_{i=1}^n \lambda_i(A^k) \leq \lambda_1(A^k) = \lambda_1(A)^k. \quad (2)$$

As in (1), the fact that  $|\operatorname{tr}(A^k)| = o(n^k)$  and that all the terms in (2) are non-positive, implies that  $|\lambda_1(A)| = o(n)$ .

We now claim that the fact that  $|\lambda_1(A)| = o(n)$  implies that  $T$  satisfies  $\mathcal{Q}$ . Suppose it does not, and let  $X, Y \subseteq V$  be two sets satisfying  $\sum_{v \in X} |d^+(v, Y) - d^-(v, Y)| = cn^2$ , for some  $c > 0$ . Let  $\mathbf{y} \in \{0, 1\}^n$  be the indicator vector for  $Y$ . We pick the vector  $\mathbf{x}$  in the following way: if  $v \notin X$ , then set the corresponding coordinate  $\mathbf{x}_v = 0$ . For  $v \in X$  such that  $d^+(v, Y) - d^-(v, Y) \geq 0$ , we set  $\mathbf{x}_v = 1$ . For all other  $v \in X$ , we set  $\mathbf{x}_v = -1$ . Now notice that for these vectors  $\mathbf{x}$  and  $\mathbf{y}$ , we have  $\mathbf{x}^T A \mathbf{y} = \sum_{v \in X} |d^+(v, Y) - d^-(v, Y)| = cn^2$ . We can normalize  $\mathbf{x}$  and  $\mathbf{y}$  to get unit vectors  $\tilde{\mathbf{x}} = \mathbf{x}/\sqrt{|X|}$  and  $\tilde{\mathbf{y}} = \mathbf{y}/\sqrt{|Y|}$  satisfying

$$\tilde{\mathbf{x}}^T A \tilde{\mathbf{y}} = (\mathbf{x}^T A \mathbf{y})/\sqrt{|X||Y|} \geq cn^2/n = cn, \quad (3)$$

where the inequality follows since  $|X|, |Y| \leq n$ . We have thus found two unit vectors  $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$  such that  $\tilde{\mathbf{x}}^T A \tilde{\mathbf{y}} \geq cn$ .

We finish the proof by showing that (3) contradicts the fact that  $|\lambda_1(A)| = o(n)$ . Let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be the orthonormal eigenvectors corresponding to the eigenvalues of  $A$ . Let  $\tilde{\mathbf{x}} = \sum_i \alpha_i \mathbf{v}_i$  and  $\tilde{\mathbf{y}} = \sum_i \beta_i \mathbf{v}_i$  be the decomposition of  $\tilde{\mathbf{x}}$  and  $\tilde{\mathbf{y}}$  along the eigenvectors (note that  $\alpha_i$  and  $\beta_i$  might be complex numbers). We have

$$\tilde{\mathbf{x}}^T A \tilde{\mathbf{y}} = \left| \sum_i \alpha_i \lambda_i(A) \beta_i \right| \leq \sqrt{\sum_i |\bar{\alpha}_i|^2 \cdot \sum_i |\lambda_i(A) \beta_i|^2} = \sqrt{\sum_i |\lambda_i(A)|^2 |\beta_i|^2} \leq |\lambda_1(A)| \quad (4)$$

where the first inequality follows by using Cauchy-Schwarz ( $\bar{\alpha}$  denotes the complex conjugate of  $\alpha$ ). We then use the fact that  $\sum_i |\alpha_i|^2 = \sum_i |\beta_i|^2 = 1$  which follow from the fact that  $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$  are unit vectors. Finally, since we have that  $|\lambda_1(A)| = o(n)$  and that  $\tilde{\mathbf{x}}^T A \tilde{\mathbf{y}} \geq cn$  equation (4) gives a contradiction. So  $T$  must satisfy  $\mathcal{Q}$ .  $\square$

We now turn to prove the second direction of Theorem 1.

**Lemma 2.4.** *Let  $k \geq 4$  be an even integer. If a tournament satisfies  $\mathcal{Q}$  then it satisfies  $\mathcal{P}(k)$ .*

*Proof.* Suppose  $T$  satisfies  $\mathcal{Q}$ . Then by the result of [5] mentioned earlier,  $T$  must also satisfy  $\mathcal{P}(4)$ . From Claim 2.2, we have that

$$|\mathrm{tr}(A^4)| = \left| \sum_{i=1}^n \lambda_i^4 \right| = o(n^4), \quad (5)$$

where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$ . We will now apply induction to show that  $|\mathrm{tr}(A^k)| = o(n^k)$  for all even integers  $k \geq 4$ . Claim 2.2 would then imply that  $\mathcal{P}(k)$  is true for all even integers  $k \geq 4$ .

Now note the following for an even integer  $k > 4$ :

$$|\mathrm{tr}(A^k)| = \left| \sum_i \lambda_i^k \right| \leq \sqrt{\sum_i \lambda_i^4 \sum_i \lambda_i^{2k-4}} \leq \sqrt{\sum_i \lambda_i^4} \cdot \left| \sum_i \lambda_i^{k-2} \right| = o(n^k).$$

The first inequality is Cauchy-Schwarz. For the second inequality, recall that by Claim 2.1 we have that  $\lambda_i^k$  are either all non-negative or non-positive. This means that  $(\sum_{i=1}^n \lambda_i^{k-2})^2 \geq \sum_{i=1}^n \lambda_i^{2k-4}$  since we lose only non-negative terms. The last equality follows by applying the induction hypothesis and (5).  $\square$

### 3 Concluding Remarks

- The proof of Lemma 2.3 shows that if  $T$  satisfies the property  $\mathcal{P}(4)$ , then  $|\lambda_1(A)| = o(n)$  which in turn implies that  $T$  satisfies  $\mathcal{Q}$ . Since we also know that  $\mathcal{Q}$  implies  $\mathcal{P}(4)$  we conclude that a tournament  $T$  is quasi-random if and only if  $|\lambda_1(A)| = o(n)$ . This is in line with other spectral characterizations of quasi-randomness for other combinatorial objects [1, 2, 3, 7, 11].
- Let  $k \geq 4$  be an even integer. Now we make an observation about  $E_k(T)$  for an arbitrary tournament  $T$  (which is not necessarily quasi-random). The total number of distinct  $k$ -cycles of  $T$  is  $\mathrm{tr}(B^k)$ , where  $B$  is the adjacency matrix of the undirected complete graph on  $n$  vertices. Since the spectrum of  $B$  is  $\{n-1, -1, \dots, -1\}$  we get  $\mathrm{tr}(B^k) = (n-1)^k + (n-1)$ . For  $k \equiv 0 \pmod{4}$ , by Claim 2.1, the eigenvalues of  $A^k$  are all non-negative and thus we have  $\mathrm{tr}(A^k) \geq 0$ . By Claim 2.2, we have that  $E_k(T) \geq ((n-1)^k + (n-1))/2$ . For  $k \equiv 2 \pmod{4}$ , we can conclude similarly using Claims 2.1 and 2.2 that  $E_k(T) \leq ((n-1)^k + (n-1))/2$ .
- We note that we can use the ideas we used in this paper to prove similar results for general directed graphs as defined by Griffiths [10]. Since the ideas required to obtain this more general result do not deviate significantly from those we have used here, we defer them to the first author's Ph.D. thesis.

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