Origin of the scaling laws of developing turbulent boundary layers

Cite as: Phys. Fluids **34**, 071402 (2022); https://doi.org/10.1063/5.0096255
Submitted: 16 April 2022 • Accepted: 21 June 2022 • Published Online: 07 July 2022

Sk Zeeshan Ali and Subhasish Dey

Physics of Fluids PERSPECTIVE scitation.org/journal/phf

Origin of the scaling laws of developing turbulent boundary layers

Cite as: Phys. Fluids **34**, 071402 (2022); doi: 10.1063/5.0096255
Submitted: 16 April 2022 · Accepted: 21 June 2022 ·
Published Online: 7 July 2022

Sk Zeeshan Ali[1,a)] and Subhasish Dey[2,3,a)]

AFFILIATIONS

[1]Department of Civil Engineering, Indian Institute of Technology Hyderabad, Telangana 502284, India
[2]Department of Civil Engineering, Indian Institute of Technology Kharagpur, West Bengal 721302, India
[3]Department of Hydraulic Engineering, State Key Laboratory of Hydro-Science and Engineering, Tsinghua University, Beijing 100084, China

Note: This paper is part of the special topic, Centennial of the Kármán-Pohlhausen Momentum-Integral Approach.
[a)]**Authors to whom correspondence should be addressed:** zeeshan@ce.iith.ac.in and sdey@iitkgp.ac.in

ABSTRACT

In this Perspective article, we seek the origin of the scaling laws of developing turbulent boundary layers over a flat plate from the perspective of the phenomenological theory of turbulence. The scaling laws of the boundary-layer thickness and the boundary shear stress in rough and smooth boundary-layer flows are established. In a rough boundary-layer flow, the boundary-layer thickness (scaled with the boundary roughness) and the boundary shear stress (scaled with the dynamic pressure) obey the "$2/(1-\sigma)$" and "$(1+\sigma)/(1-\sigma)$" scaling laws, respectively, with the streamwise distance (scaled with the boundary roughness). Here, σ is the spectral exponent. In a smooth boundary-layer flow, the boundary-layer thickness (scaled with the viscous length scale) and the boundary shear stress (scaled with the dynamic pressure) obey the "$8/(5-3\sigma)$" and "$3(1+\sigma)/(5-3\sigma)$" scaling laws, respectively, with the Reynolds number characterized by the streamwise distance.

I. INTRODUCTION

After Prandtl,[1] the boundary layer is recognized as the near-boundary fluid layer, which is affected by the boundary shear. The boundary-layer thickness δ is measured as the distance z from the boundary, where the local streamwise velocity u attains 99% of the free stream velocity U. When a flat plate is aligned parallelly to an approach free stream, the boundary-layer thickness δ grows with the streamwise distance x (Fig. 1), displaying a switchover from a laminar boundary layer to a turbulent boundary layer via a transition region.

Following Prandtl's discovery of the boundary layer and Blasius' similarity solution,[2] a significant advancement in the boundary-layer theory is the momentum-integral approach put forward by Theodore von Kármán and Karl Pohlhausen.[3,4] The applications of the momentum-integral approach are far-reaching in a variety of fluid mechanics problems.[5,6]

In the momentum-integral approach, the boundary shear stress τ_0 follows the generalized von Kármán momentum-integral equation

$$\tau_0(x) = \rho U \delta^* \frac{\mathrm{d}U}{\mathrm{d}x} + \rho \frac{\mathrm{d}}{\mathrm{d}x}(U^2\theta), \tag{1}$$

where ρ is the mass density of fluid, δ^* is the displacement thickness, and θ is the momentum thickness. The δ^* and θ are expressed, respectively, as

$$\delta^* = \int_0^{\delta(x)} \left(1 - \frac{u}{U}\right)\mathrm{d}z, \tag{2a}$$

$$\theta = \int_0^{\delta(x)} \frac{u}{U}\left(1 - \frac{u}{U}\right)\mathrm{d}z. \tag{2b}$$

For a zero-pressure gradient flow, $\mathrm{d}U/\mathrm{d}x = 0$. Therefore, Eq. (1) reduces to

$$\tau_0(x) = \rho U^2 \frac{\mathrm{d}\theta}{\mathrm{d}x}. \tag{3}$$

Herein, we focus primarily on the developing turbulent boundary layer. To solve Eq. (3), accurate description of the velocity distribution, e.g., the law of the wall,[7] within the boundary layer is an essential prerequisite. Conventionally, in smooth boundary-layer flow with a zero-pressure gradient, the von Kármán momentum-integral equation, Eq. (3), is solved by considering a 1/7-th power law of self-similar velocity distribution within the boundary layer.[5] In addition, the Blasius

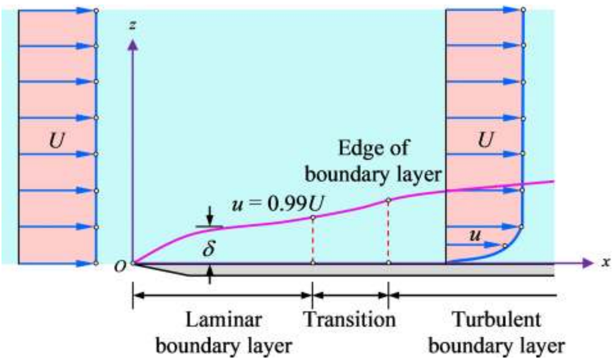

FIG. 1. Sketch of a developing boundary layer over a flat plate. Here, u is the local streamwise velocity, U is the free stream velocity, and δ is the boundary-layer thickness.

formula for the boundary shear stress is sought in order to obtain the development of the boundary-layer thickness and the boundary shear stress (for detailed derivation, see Appendix). While the von Kármán momentum-integral equation has a strong theoretical background, the application of the Blasius formula to a developing turbulent boundary-layer flow is purely empirical. Therefore, the classical scaling laws of the boundary-layer thickness and the boundary shear stress in standard textbooks of fluid mechanics lack a complete theoretical foundation. Moreover, the Blasius formula is strictly valid for a smooth boundary-layer flow. As a result, the classical expressions, being applicable to a smooth boundary-layer flow, do not reflect the role of boundary roughness in the scaling laws of the boundary-layer thickness and the boundary shear stress. However, it is worth mentioning that for a velocity distribution, an equivalence of the $1/m$-th power law and the logarithmic law produces $m = \kappa(8/\lambda)^{1/2}$,[8] where κ is the von Kármán constant and λ is the Darcy–Weisbach friction factor. It indicates that through the Darcy–Weisbach friction factor, the boundary roughness is implicitly linked with the exponent m, although m is traditionally set as a constant value 7. Therefore, in a rough boundary-layer flow, the scaling laws of the boundary-layer thickness and the boundary shear stress remain unexplored.

In this Perspective article, we aim at finding the origin of the scaling laws of developing turbulent boundary layers in both rough and smooth boundary-layer flows from the phenomenological theory of turbulence. The phenomenological theory of turbulence stems from Richardson's rhyming verse on the disintegration of a large turbulent eddy into smaller eddies through an energy cascade.[9]

This article is organized as follows. The scaling of the boundary-layer thickness and the boundary shear stress is given in Sec. II. The phenomenological framework is developed in Sec. III. The scaling laws of rough and smooth boundary-layer flows are deduced in Secs. IV and V, respectively. Finally, conclusions are drawn in Sec. VI.

II. SCALING OF BOUNDARY-LAYER THICKNESS AND BOUNDARY SHEAR STRESS

The dimensional analysis enables us to express the scaled boundary-layer thickness and the boundary shear stress in functional forms. In a rough boundary-layer flow, the boundary-layer thickness δ can be scaled with the boundary roughness k_s, whereas in a smooth boundary-layer flow, δ can be scaled with the viscous length scale υ/U. Here, υ is the coefficient of kinematic viscosity of fluid. On the contrary, in both rough and smooth boundary-layer flows, the boundary shear stress τ_0 can be scaled with the dynamic pressure $\rho U^2/2$.

In a rough boundary-layer flow, the scaled boundary-layer thickness and the boundary shear stress are expressed as follows:

$$\hat{\delta}|_{\text{rough}}\left(\equiv \frac{\delta}{k_s}\right) = f(\hat{x}), \tag{4}$$

$$\hat{\tau}_0|_{\text{rough}}\left(\equiv \frac{2\tau_0}{\rho U^2}\right) = f(\hat{x}), \tag{5}$$

where $\hat{x}$ is the streamwise distance scaled with the boundary roughness ($\equiv x/k_s$).

On the contrary, in a smooth boundary-layer flow, the scaled boundary-layer thickness and the boundary shear stress are expressed as follows:

$$\hat{\delta}|_{\text{smooth}}\left(\equiv \frac{U\delta}{\upsilon}\right) = f(R_x), \tag{6}$$

$$\hat{\tau}_0|_{\text{smooth}}\left(\equiv \frac{2\tau_0}{\rho U^2}\right) = f(R_x), \tag{7}$$

where R_x is the Reynolds number characterized by the streamwise distance ($\equiv Ux/\upsilon$).

Herein, the appropriate form of the scaling relationships in Eqs. (4)–(7) and their origin are explored from the phenomenological theory of turbulence.

III. PHENOMENOLOGICAL FRAMEWORK

After Richardson,[9] Kolmogorov's theory[10] laid down the foundation of the phenomenological theory of turbulence.[11] The implication of the phenomenological theory is to predict the scaling laws of a problem in a scientifically simpler way requiring no empirical influences. A variety of applications of the phenomenological theory of turbulence can be found in literature. To be specific, the applications include derivations of the laws of the wall shear flow,[12–17] scaling laws of the rough open-channel flow,[18,19] friction law of the permeable-wall flow,[20] and origin of the scaling laws in fluvial systems.[21–26] A detailed review on this topic was reported elsewhere.[27]

In this section, we aim at deriving a scaling law for the boundary shear stress τ_0 developed in a turbulent boundary layer flow, having the boundary-layer thickness δ and the boundary roughness k_s (Fig. 2). The boundary shear stress is caused by the turbulent eddies as a result of momentum transfer. According to the phenomenological theory of turbulence, a turbulent flow comprises eddies having wide-ranging length scales. Due to the mean flow instability, the large eddies of length scale L become unstable. They eventually break down by distributing the turbulent kinetic energy (TKE) into smaller eddies, which undergo the similar breakdown process to produce even smaller eddies. This process goes on until the length scale becomes equal to Kolmogorov's length scale η for which the fluid molecular viscosity dissipates the TKE into heat. The TKE per unit mass of a turbulent eddy having a characteristic length scale l is expressed as[24]

$$v_l^2 = \int_{1/l}^{\infty} E(k)\mathrm{d}k, \tag{8}$$

where v_l is the characteristic velocity of the turbulent eddy, $E(k)$ is the energy spectrum, and k is the wavenumber.

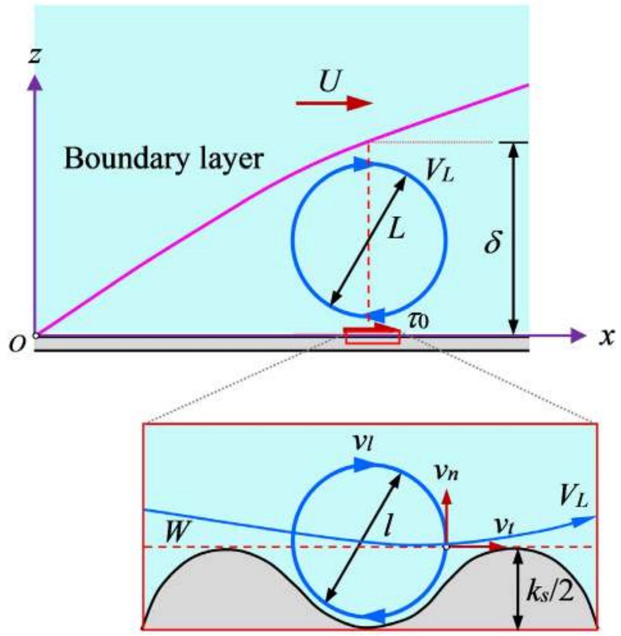

FIG. 2. Phenomenological sketch of the developing turbulent boundary layer with a boundary-layer thickness δ and a free stream velocity U. Here, τ_0 is the boundary shear stress, and L and V_L are the length and velocity scales of a large turbulent eddy, respectively. The enlarged view shows the interaction of a local turbulent eddy, having a length scale l and a velocity scale v_l, and the boundary roughness k_s. The wetted surface W (shown by the horizontal broken line) is tangential to the boundary roughness summits, and v_t and v_n are the tangential and normal velocity fluctuations, respectively.

The $E(k)$ in the inertial subrange takes the following form:[11]

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}, \tag{9}$$

where ε is the TKE dissipation rate and the symbol "$\sim$" signifies the "scales with." The ε associated with the large eddies can be scaled as

$$\varepsilon \sim \frac{V_L^2}{t_L} \sim \frac{V_L^3}{L}, \tag{10}$$

where V_L is the characteristic velocity of large eddies and t_L is the eddy turnover time, i.e., $t_L \sim L/V_L$. With the above scaling, the $E(k)$ in Eq. (9) becomes

$$E(k) \sim V_L^2 L (kL)^{-5/3}. \tag{11}$$

The above form of $E(k)$ is substituted in Eq. (8). The result is

$$v_l \sim V_L \left(\frac{l}{L}\right)^{1/3}. \tag{12}$$

The above scaling law reveals the link between the ratio of the velocity scales and the ratio of the length scales. With regard to the present problem, the V_L in Eq. (12) can be scaled with the free stream velocity U, while L can be scaled with the boundary-layer thickness $\delta(x)$. Therefore

$$V_L \sim U \neq f(x) \quad \text{and} \quad L \sim \delta(x). \tag{13}$$

Equation (13) allows us to express Eq. (12) in the following form:

$$v_l \sim U \left[\frac{l}{\delta(x)}\right]^{1/3}. \tag{14}$$

However, the energy spectrum in Eq. (11) can be expressed in a more general form as follows:

$$E(k) \sim V_L^2 L^{1+\sigma} k^{\sigma}, \tag{15}$$

where σ represents the spectral exponent. For instance, $\sigma = -5/3$ holds for a three-dimensional turbulence, whereas $\sigma = -3$ is valid for a two-dimensional turbulence.[24] However, as far as the motivation of this perspective article is concerned, we focus primarily on the scaling laws for a three-dimensional turbulence. With the above form of $E(k)$ and performing similar exercise, the general form of Eq. (14) is expressed as

$$v_l \sim U \left[\frac{l}{\delta(x)}\right]^{-(1+\sigma)/2}. \tag{16}$$

The Reynolds shear stress τ, caused by a turbulent eddy of length scale l and velocity scale v_l (Fig. 2), acting at the wetted surface W tangential to the boundary roughness summits is

$$\tau = \rho \overline{v_t v_n}, \tag{17}$$

where v_t and v_n are the tangential and normal velocity fluctuations, respectively, and the overbar stands for the time-averaging operator. Below the wetted surface W, the flow velocity is quite small. Therefore, the approach flow transmits a trivial momentum per unit volume tangential to the W. By contrast, above the W, the flow velocity is finite and scales with the V_L. Therefore, the approach flow transmits a significant momentum per unit volume ($\sim \rho V_L$) tangential to the W. The Reynolds shear stress τ is, thus, expressed as the product of the net momentum contrast across the W and the normal component of the eddy turnover velocity.[12,18] It follows

$$\tau \sim \rho V_L v_n. \tag{18}$$

Since the length scale l of the turbulent eddy spans over a significant range, it is important to identify the foremost eddies causing the momentum transfer at the wetted surface W. It is clear that the eddies of length scale l larger than the boundary roughness k_s contribute weakly to the momentum transfer because they are unable to bestride the surface W completely. On the contrary, the eddies of length scale l smaller than k_s perfectly bestride the surface W, offering a substantial normal component of the eddy turnover velocity.[18] Therefore, the foremost eddies possess a length scale $l = k_s$ (Fig. 2). As the foremost eddies contribute their characteristic velocity v_l normal to the W, the v_n can be scaled as $v_n \sim v_l$. With $V_L \sim U$ [Eq. (13)] and using Eq. (16), the Reynolds shear stress τ at the W can be obtained from Eq. (18). In the absence of the viscous shear stress, the Reynolds shear stress τ developed at the wetted surface W is balanced by the boundary shear stress τ_0. Therefore, the τ_0 is expressed as

$$\tau_0(x) \sim \rho U^2 \left[\frac{l}{\delta(x)}\right]^{-(1+\sigma)/2} \sim \rho U^2 \left[\frac{k_s}{\delta(x)}\right]^{-(1+\sigma)/2}. \tag{19}$$

Equations (3) and (19) are two independent expressions for the boundary shear stress. The former is the classical expression, while the latter is the phenomenological expression. Equation (3) can be further simplified by setting a self-similar velocity distribution within the boundary layer. The self-similarity implies

$$\frac{u}{U} \equiv F(\zeta) \quad \text{with} \quad \zeta \equiv \frac{z}{\delta(x)}. \tag{20}$$

With Eq. (20), the momentum thickness θ from Eq. (2b) is expressed as

$$\theta = c\delta(x) \quad \text{with} \quad c = \int_0^1 F(1-F)\mathrm{d}\zeta. \tag{21}$$

In the above, c is a constant, whose value is immaterial in the present context. In fact, it depends on the particular form of velocity distribution within the boundary layer. Therefore, Eq. (21) suggests that the momentum thickness θ can be scaled as $\theta \sim \delta(x)$. Therefore, Eq. (3) can be expressed as

$$\tau_0(x) \sim \rho U^2 \frac{\mathrm{d}\delta(x)}{\mathrm{d}x}. \tag{22}$$

IV. SCALING LAWS OF ROUGH BOUNDARY-LAYER FLOW

Equations (19) and (22) can be solved together to obtain the scaling laws of the boundary-layer thickness and the boundary shear stress. Combining Eqs. (19) and (22) yields the following differential equation:

$$[\delta(x)]^{-(1+\sigma)/2}\mathrm{d}\delta \sim k_s^{-(1+\sigma)/2}\mathrm{d}x. \tag{23}$$

Integrating the above equation subject to the boundary condition $\delta(x=0)=0$ provides the scaling law of the boundary-layer thickness as

$$\delta(x) \sim k_s^{-(1+\sigma)/(1-\sigma)} x^{2/(1-\sigma)} \Rightarrow \hat{\delta}|_{\text{rough}} \sim \hat{x}^{2/(1-\sigma)}. \tag{24}$$

Substituting Eq. (24) into Eq. (22) results the scaling law of the boundary shear stress as

$$\tau_0(x) \sim \rho U^2 k_s^{-(1+\sigma)/(1-\sigma)} x^{(1+\sigma)/(1-\sigma)} \Rightarrow \hat{\tau}_0|_{\text{rough}} \sim \hat{x}^{(1+\sigma)/(1-\sigma)}. \tag{25}$$

Therefore, in a rough boundary-layer flow, the *scaling laws* of the boundary-layer thickness [Eq. (24)] and the boundary shear stress [Eq. (25)] together state:

> *In a rough boundary-layer flow, the boundary-layer thickness (scaled with the boundary roughness) and the boundary shear stress (scaled with the dynamic pressure) obey the '2/(1−σ)' and '(1+σ)/(1−σ)' scaling laws, respectively, with the streamwise distance (scaled with the boundary roughness), where σ is the spectral exponent.*

For a three-dimensional turbulence, $\sigma = -5/3$ for which Eqs. (24) and (25) yield the scaling laws of the boundary-layer thickness and the boundary shear stress in a rough boundary-layer flow as follows:

$$\delta(x) \sim k_s^{1/4} x^{3/4} \Rightarrow \hat{\delta}|_{\text{rough}} \sim \hat{x}^{3/4}, \tag{26}$$

$$\tau_0(x) \sim \rho U^2 k_s^{1/4} x^{-1/4} \Rightarrow \hat{\tau}_0|_{\text{rough}} \sim \hat{x}^{-1/4}. \tag{27}$$

Equations (26) and (27) show the explicit functional form of Eqs. (4) and (5), respectively. It is interesting to note that Eq. (19) for $\sigma = -5/3$ corresponds to Strickler's scaling law.

V. SCALING LAWS OF SMOOTH BOUNDARY-LAYER FLOW

In a smooth boundary-layer flow, the boundary roughness $k_s \to 0$. Therefore, it appears from Eq. (19) that the boundary shear stress vanishes as the boundary roughness approaches zero. However, this contradicts the experimental observations and, thus, is not physically acceptable. Importantly, in a smooth boundary-layer flow, the boundary roughness is protected by a thin viscous sublayer, whose thickness is considered to be five times the Kolmogorov length scale ($= 5\eta$). The η is expressed as

$$\eta = \left(\frac{\upsilon^3}{\varepsilon}\right)^{1/4}. \tag{28}$$

Using Eqs. (10) and (13), the above equation becomes

$$\eta \sim \frac{\upsilon^{3/4}[\delta(x)]^{1/4}}{U^{3/4}}. \tag{29}$$

Therefore, the foremost eddy causing the boundary shear stress in a smooth boundary-layer flow is of the length scale $l = 5\eta$.[18] From Eq. (19), the boundary shear stress τ_0 is expressed as

$$\tau_0(x) \sim \rho U^2 \left[\frac{\eta}{\delta(x)}\right]^{-(1+\sigma)/2}. \tag{30}$$

Substituting Eq. (29) into Eq. (30) yields

$$\tau_0(x) \sim \rho U^2 \left[\frac{\upsilon}{U\delta(x)}\right]^{-3(1+\sigma)/8}. \tag{31}$$

Equations (22) and (31) can be solved together to obtain the scaling laws of the boundary-layer thickness and the boundary shear stress. Combining Eqs. (22) and (31) yields the following differential equation:

$$[\delta(x)]^{-3(1+\sigma)/8}\mathrm{d}\delta \sim \left(\frac{\upsilon}{U}\right)^{-3(1+\sigma)/8}\mathrm{d}x. \tag{32}$$

Integrating the above equation subject to the boundary condition $\delta(x=0)=0$ provides the scaling law of the boundary-layer thickness as

$$\delta(x) \sim \left(\frac{\upsilon}{U}\right)^{-3(1+\sigma)/(5-3\sigma)} x^{8/(5-3\sigma)} \Rightarrow \hat{\delta}|_{\text{smooth}} \sim R_x^{8/(5-3\sigma)}. \tag{33}$$

Substituting Eq. (33) into Eq. (22) results the scaling law of the boundary shear stress as

$$\tau_0(x) \sim \rho U^2 \left(\frac{\upsilon}{U}\right)^{-3(1+\sigma)/(5-3\sigma)} x^{3(1+\sigma)/(5-3\sigma)} \Rightarrow \hat{\tau}_0|_{\text{smooth}} \sim R_x^{3(1+\sigma)/(5-3\sigma)}. \tag{34}$$

Therefore, in a smooth boundary-layer flow, the *scaling laws* of the boundary-layer thickness [Eq. (33)] and the boundary shear stress [Eq. (34)] together state:

> *In a smooth boundary-layer flow, the boundary-layer thickness (scaled with the viscous length scale) and the boundary shear stress (scaled with the dynamic pressure) obey the "8/(5 − 3σ)" and "3(1+σ)/(5 − 3σ)" scaling laws, respectively, with the Reynolds number characterized by the streamwise distance, where σ is the spectral exponent.*

For a three-dimensional turbulence, $\sigma = -5/3$ for which Eqs. (33) and (34) yield the scaling laws of the boundary-layer thickness and the boundary shear stress in a smooth boundary-layer flow as follows:

$$\delta(x) \sim \left(\frac{\upsilon}{U}\right)^{1/5} x^{4/5} \Rightarrow \hat{\delta}|_{\text{smooth}} \sim R_x^{4/5}, \tag{35}$$

$$\tau_0(x) \sim \rho U^2 \left(\frac{\upsilon}{U}\right)^{1/5} x^{-1/5} \Rightarrow \hat{\tau}_0|_{\text{smooth}} \sim R_x^{-1/5}. \tag{36}$$

Equations (35) and (36) show the explicit functional form of Eqs. (6) and (7), respectively. The above scaling laws match completely with the classical expressions for the boundary-layer thickness and the boundary shear stress in a smooth boundary-layer flow, as given in standard textbooks (see Appendix). It is interesting to note that Eq. (31) for $\sigma = -5/3$ corresponds to Blasius' scaling law.

VI. CONCLUSION

This Perspective article, dedicated to the *Centennial of the Kármán-Pohlhausen Momentum-Integral Approach*, presents the origin of the scaling laws of developing turbulent boundary layers. The phenomenological theory of turbulence is applied to explore the scaling laws of the boundary-layer thickness and the boundary shear stress in both rough and smooth boundary-layer flows. In a rough boundary-layer flow, the boundary-layer thickness (scaled with the boundary roughness) and the boundary shear stress (scaled with the dynamic pressure) obey unique scaling laws with the streamwise distance (scaled with the boundary roughness). On the contrary, in a smooth boundary-layer flow, the boundary-layer thickness (scaled with the viscous length scale) and the boundary shear stress (scaled with the dynamic pressure) obey unique scaling laws with the Reynolds number characterized by the streamwise distance. The scaling relationships reflect the role of the spectral exponent. To be specific, for a three-dimensional turbulence, the boundary-layer thickness (scaled with the boundary roughness) and the boundary shear stress (scaled with the dynamic pressure) in a rough boundary-layer flow obey the "3/4" and "−1/4" scaling laws with the streamwise distance (scaled with the boundary roughness). On the contrary, the boundary-layer thickness (scaled with the viscous length scale) and the boundary shear stress (scaled with the dynamic pressure) in a smooth boundary-layer flow obey the "4/5" and "−1/5" scaling laws with the Reynolds number characterized by the streamwise distance.

ACKNOWLEDGMENTS

S.Z.A. acknowledges the Institute Seed Grant of Indian Institute of Technology Hyderabad in carrying out this research. S.D. acknowledges the J C Bose Fellowship Award [Funded by DST | Science and Engineering Research Board (SERB), Grant Reference No. JCB/2018/000004] in pursuing this work.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Sk Zeeshan Ali: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Resources (equal); Validation (equal); Visualization (equal); Writing – original draft (lead); Writing – review and editing (equal). **Subhasish Dey:** Conceptualization (equal); Data curation (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Resources (equal); Validation (equal); Visualization (equal); Supervision (lead); Project administration (lead); Writing – original draft (supporting); Writing – review and editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX: CLASSICAL EXPRESSIONS FOR $\delta(X)$ AND $\tau_0(X)$ IN A SMOOTH BOUNDARY-LAYER FLOW

In a smooth boundary-layer flow, a 1/7-th power law of velocity distribution is traditionally used. Therefore

$$\frac{u}{U} = F(\zeta) \quad \text{with} \quad F(0 \le \zeta \le 1) = \zeta^{1/7}. \tag{A1}$$

Inserting the above equation into Eq. (3) produces

$$\tau_0(x) = \frac{7}{72}\rho U^2 \frac{\mathrm{d}\delta(x)}{\mathrm{d}x}. \tag{A2}$$

Moreover, the boundary shear stress in a smooth turbulent boundary-layer flow follows the Blasius formula:

$$\tau_0(x) = 2.28 \times 10^{-2} \rho U^2 \left[\frac{\upsilon}{U\delta(x)}\right]^{1/4}. \tag{A3}$$

Combining Eqs. (A2) and (A3) yields a differential equation of $\delta(x)$, which is solved subject to the boundary condition $\delta(x=0)=0$. The result is

$$\delta(x) = 0.376\left(\frac{\upsilon}{U}\right)^{1/5} x^{4/5} \Rightarrow \hat{\delta}|_{\text{smooth}} = 0.376 R_x^{4/5}. \tag{A4}$$

Substituting Eq. (A4) into Eq. (A2) yields the boundary shear stress as

$$\tau_0(x) = 2.91 \times 10^{-2} \rho U^2 \left(\frac{\upsilon}{U}\right)^{1/5} x^{-1/5}$$
$$\Rightarrow \hat{\tau}_0|_{\text{smooth}} = 2.91 \times 10^{-2} R_x^{-1/5}. \tag{A5}$$

Equations (A4) and (A5) corroborate the scaling laws, as deduced in Eqs. (35) and (36).

REFERENCES

[1] L. Prandtl, "Über flüssigkeitsbewegung bei sehr kleiner reibung," in *Verhandlungen Des III* (Internationalen Mathematiker Kongresses, 1904), pp. 484–491.
[2] H. Blasius, "Grenzschichten in flüssigkeiten mit kleiner reibung," J. Appl. Math. Phys. **56**, 1–37 (1908).
[3] T. von Kármán, "Über laminare und turbulente reibung," Z. Angew. Math. Mech. **1**, 233–252 (1921).
[4] K. Pohlhausen, "Zur näherungsweisen integration der differentialgleichung der laminaren grenzschicht," Z. Angew. Math. Mech. **1**, 252–290 (1921).
[5] H. Schlichting, *Boundary-Layer Theory* (McGraw-Hill, New York, 1979).
[6] J. Majdalani and L.-J. Xuan, "On the Kármán momentum-integral approach and the Pohlhausen paradox," Phys. Fluids **32**(12), 123605 (2020).
[7] S. Z. Ali and S. Dey, "The law of the wall: A new perspective," Phys. Fluids **32**(12), 121401 (2020).
[8] S. Dey, *Fluvial Hydrodynamics: Hydrodynamic and Sediment Transport Phenomena* (Springer-Verlag, Berlin, Germany, 2014).
[9] L. F. Richardson, *Weather Prediction by Numerical Process* (Cambridge University Press, Cambridge, 1922).
[10] A. N. Kolmogorov, "The local structure of turbulence in incompressible viscous fluids at very large Reynolds numbers," Dokl. Akad. Nauk SSSR **30**, 299–303 (1941).
[11] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[12] G. Gioia and P. Chakraborty, "Turbulent friction in rough pipes and the energy spectrum of the phenomenological theory," Phys. Rev. Lett. **96**(4), 044502 (2006).
[13] G. Gioia, P. Chakraborty, and F. A. Bombardelli, "Rough-pipe flows and the existence of fully developed turbulence," Phys. Fluids **18**(3), 038107 (2006).
[14] G. Gioia, N. Guttenberg, N. Goldenfeld, and P. Chakraborty, "Spectral theory of the turbulent mean-velocity profile," Phys. Rev. Lett. **105**(18), 184501 (2010).
[15] G. G. Katul, A. Porporato, C. Manes, and C. Meneveau, "Co-spectrum and mean velocity in turbulent boundary layers," Phys. Fluids **25**(9), 091702 (2013).
[16] G. G. Katul and C. Manes, "Cospectral budget of turbulence explains the bulk properties of smooth pipe flow," Phys. Rev. E **90**(6), 063008 (2014).
[17] G. Gioia and P. Chakraborty, "Spectral derivation of the classic laws of wall-bounded turbulent flows," Proc. R Soc. A **473**(2204), 20170354 (2017).
[18] G. Gioia and F. A. Bombardelli, "Scaling and similarity in rough channel flows," Phys. Rev. Lett. **88**(1), 014501 (2001).
[19] S. Bonetti, G. Manoli, C. Manes, A. Porporato, and G. G. Katul, "Manning's formula and Strickler's scaling explained by a co-spectral budget model," J. Fluid Mech. **812**, 1189–1212 (2017).
[20] C. Manes, L. Ridolfi, and G. Katul, "A phenomenological model to describe turbulent friction in permeable-wall flows," Geophys. Res. Lett. **39**(14), L14403, https://doi.org/10.1029/2012GL052369 (2012).
[21] G. Gioia and F. A. Bombardelli, "Localized turbulent flows on scouring granular beds," Phys. Rev. Lett. **95**(1), 014501 (2005).
[22] F. A. Bombardelli and G. Gioia, "Scouring of granular beds by jet-driven axisymmetric turbulent cauldrons," Phys. Fluids **18**(8), 088101 (2006).
[23] C. Manes and M. Brocchini, "Local scour around structures and the phenomenology of turbulence," J. Fluid Mech. **779**(Sept), 309–324 (2015).
[24] S. Z. Ali and S. Dey, "Origin of the scaling laws of sediment transport," Proc. R. Soc. A **473**(2197), 20160785 (2017).
[25] S. Dey and S. Z. Ali, "Origin of the onset of meandering of a straight river," Proc. R. Soc. A **473**(2204), 20170376 (2017).
[26] F. A. Bombardelli, M. Palermo, and S. Pagliara, "Temporal evolution of jet induced scour depth in cohesionless granular beds and the phenomenological theory of turbulence," Phys. Fluids **30**(8), 085109 (2018).
[27] S. Z. Ali and S. Dey, "Impact of phenomenological theory of turbulence on pragmatic approach to fluvial hydraulics," Phys. Fluids **30**(4), 045105 (2018).