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Letter

# On time independent Schrödinger equations in quantum mechanics by the homotopy analysis method

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ABSTRACT

A general analytic approach, namely the homotopy analysis method (HAM), is applied to solve the time independent Schrödinger equations. Unlike perturbation method, the HAM-based approach does not depend on any small physical parameters, corresponding to small disturbances. Especially, it provides a convenient way to gain the convergent series solution of quantum mechanics. This study illustrates the advantages of this HAM-based approach over the traditional perturbative approach, and its general validity for the Schrödinger equations. Note that perturbation methods are widely used in quantum mechanics, but perturbation results are hardly convergent. This study suggests that the HAM might provide us a new, powerful alternative to gain convergent series solution for some complicated problems in quantum mechanics, including many-body problems, which can be directly compared with the experiment data to improve the accuracy of the experimental findings and/or physical theories.



In quantum mechanics, the Schrödinger equations [1-7] govern the most interesting problems, such as harmonic and anharmonic oscillators, Morse and Pöschl-Teller potentials, etc. The perturbation methods have been developed and broadly applied to solve many problems in quantum mechanics and quantum chromodynamics (QCD) [1-9]. However, in most cases, accurate approximations can be gained only for physical quantities in a rather small range. The perturbation methods depend on a small physical parameter, i.e. perturbed parameter related to disturbances. In general, perturbation methods fail to give a accurate enough solution for large disturbances far from a known status. So do other traditional analytic approximation methods. In practices, a perturbation result is often used to check experimental data without verifying the convergence of the perturbation series solution. So, it is important to provide convergent results for large enough disturbances, which can be directly used to verify experimental data.

To overcome the limitations of perturbation methods and other traditional analytic methods, a general analytical approach, namely the homotopy analysis method (HAM), was proposed by Liao [10-12] using homotopy, a basic concept in topology. Unlike the perturbation methods, the HAM can solve a nonlinear equation without considering any small/large physical variables/parameters. Especially, the HAM gives a straightforward way to guarantee the convergence of solution series, and due to this, accurate approximations are always obtained even for strongly nonlinear equations. Besides, unlike all other analytical approximation methods, the HAM has a great freedom to choose the auxiliary linear operator and the initial guess of unknowns. The HAM has been widely applied to solve many strongly non-linear equations in different areas [13-18]. Note that the HAM has been used to gain new solutions of some highly nonlinear problems, which can not be obtained by other analytic methods even as well as numerical methods [12]. All of these illustrate the validity, novelty value and superiority of the HAM over all other analytic approximation methods.

Recently, a HAM-based approach was proposed by Liao [18] to solve the time independent Schrödinger equation in quantum mechanics, i.e.

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$$H\psi_n(x) = E_n\psi_n(x), \quad n = 0, 1, \ldots, \tag{1}$$

where $H$ is a Hamiltonian operator, $E_n$ and $\psi_n(x)$ are the unknown eigenvalue and the eigenfunction of $H$, respectively. Here, we assume that each eigenvalue $E_n$ corresponds to an unique eigenfunction $\psi_n(x)$. The anharmonic oscillator was used by Liao [18] as an example to illustrate the validity of this approach.

In this manuscript, two examples in quantum mechanics are further used to illustrate the general validity of the HAM-based approach proposed by Liao [18]. The mathematical formulas of this HAM-based approach are briefly described, then we show two examples to verify the validity of this approach.

Liao [18] assumed that the eigenfunction $\psi_n(x)$ can be expressed by the set of known eigenfunction $\psi_k^b(x)$, satisfying

$$H_0\psi_k^b(x) = E_k^b\psi_k^b(x), \tag{2}$$

where $H_0$ is a known Hamiltonian operator, $E_k^b$ and $\psi_k^b(x)$ are the corresponding known eigenvalue and the eigenfunction of $H_0$, respectively. Assume that each eigenvalue $E_k^b$ corresponds to an unique eigenfunction $\psi_k^b(x)$, and besides $\psi_k^b(x)$ is orthonormal, i.e.

$$\left(\psi_m^b, \psi_n^{b*}\right) = \begin{cases} 1, & \text{when } m = n, \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where $\psi_n^{b*}$ is a complex conjugate of $\psi_n^b$.

The homotopy series solution of the time independent Schrödinger equation read

$$\psi_n(x) = \psi_n^{(0)}(x) + \sum_{m=1}^{+\infty} \psi_n^{(m)}(x), \tag{4}$$

$$E_n = E_n^{(0)} + \sum_{m=1}^{+\infty} E_n^{(m)}, \tag{5}$$

where

$$E_n^{(k-1)} = \frac{\left(H\psi_n^{(k-1)} - \sum_{i=0}^{k-2} E_n^{(i)}\psi_n^{(k-1-i)},\ \psi_n^{b*}\right)}{\left(\psi_n^{(0)},\ \psi_n^{b*}\right)}, \tag{6}$$

$$\psi_n^{(k)} = \chi_k\,\psi_n^{(k-1)} + \sum_{m=0}^{N} a_{n,m}^{(k)}\,\psi_m^b \tag{7}$$

with $N$ being a large enough truncation number and

$$a_{n,m}^{(k)} = c_0\left(\frac{\Delta_{k-1}^{n,m}}{E_m^b - E_n^b}\right), \quad \text{when } m \neq n, \tag{8}$$

$$a_{n,n}^{(k)} = -\frac{\left(\left(H - \sum_{j=0}^{k-1} E_n^{(j)}\right)\hat{\psi}_n^{(k)}, \left(H - \sum_{j=0}^{k-1} E_n^{(j)}\right)\psi_n^{b*}\right)}{\left(\left(H - \sum_{j=0}^{k-1} E_n^{(j)}\right)\psi_n^b, \left(H - \sum_{j=0}^{k-1} E_n^{(j)}\right)\psi_n^{b*}\right)}, \tag{9}$$

in which

$$\Delta_i^{n,m} = \left(H\,\psi_n^{(i)} - \sum_{j=0}^{i} E_n^{(j)}\,\psi_n^{(i-j)}, \psi_m^{b*}\right), \tag{10}$$

$$\hat{\psi}_n^{(k)} = \sum_{i=0}^{k-1}\psi_n^{(i)} + \chi_k\,\psi_n^{(k-1)} + \sum_{m=0, m\neq n}^{N} a_{n,m}^{(k)}\,\psi_m^b(x), k \geq 1, \tag{11}$$

and

$$\chi_k = \begin{cases} 0, & \text{when } k \leq 1, \\ 1, & \text{otherwise.} \end{cases} \tag{12}$$

Especially, $c_0$ is the so-called "convergence-control parameter" in the frame of the HAM, which can guarantee the convergence of solution series. The optimal value of the $c_0$ is determined by means of the minimum of the residual error square of the governing Eq. (1). This is quite different from all other analytic approximation methods. In fact, it is the so-called "convergence-control parameter" $c_0$ that distinguishes the HAM from all of others. For details, please refer to Liao [18].

At the $M$th-order of approximation, we have

$$\psi_n(x) \approx \psi_n^{(0)}(x) + \sum_{m=1}^{M}\psi_n^{(m)}(x), \tag{13}$$

$$E_n \approx E_n^{(0)} + \sum_{m=1}^{M} E_n^{(m)}. \tag{14}$$

Unlike perturbation methods, the HAM provides us the great flexibility to choose the initial guess $\psi_n^0(x)$. Using this freedom, we can first choose $\psi_n^{(0)} = \psi_n^b$ and then gain a $M$th-order approximation of $\psi_n$ and $E_n$ by means of the above-mentioned formulas. Then, the $M$th-order approximation of $\psi_n$ can be further used as an initial guess $\psi_n^{(0)}$ to gain a better $M$th-order approximation of $\psi_n$ and $E_n$ in a similar way. In fact, this provides us a iteration HAM-based approach for the considered problem, which can accelerate the convergence greatly, as illustrated by Liao [18].

In this study, we further consider two examples in the quantum mechanics, the complex cubic anharmonic oscillator and the perturbed Pöschl-Teller potential, to show the general

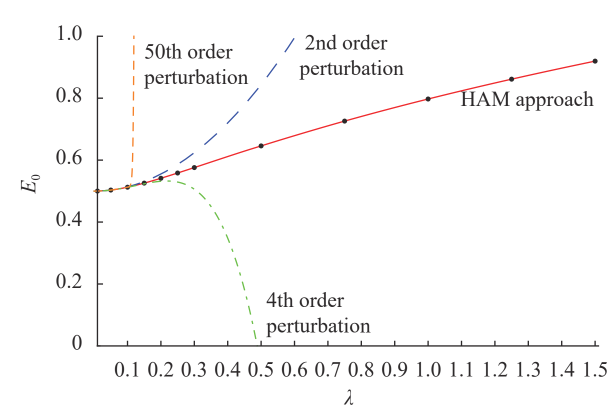


**Fig. 1.** The analytic approximations of the eigenvalue $E_0$ of Eq. (15) versus $\lambda$. Solid line: convergent results given by the HAM-based approach; long dashed line: 2nd-order perturbative approach; dashed-dotted line: 4th-order perturbative approach; dashed line: 50th-order perturbative approach.

**Table 1** The analytic approximations of the eigenvalue $E_0$ and residual error square of Eq. (15) using 6th-order HAM-based iteration approach at different times of iteration $(i)$ in case of $\lambda = 0.15$ with $c_0 = -0.15$, $N = 45$ and $\lambda = 0.2$ with $c_0 = -0.1$, $N = 50$, compared by the perturbation approach at different order of approximation $(M)$.

| $\lambda$ | $i$ or $M$ | $E_0$ (HAM) | error (HAM) | $E_0$ (perturbation) | error (perturbation) |
|---|---|---|---|---|---|
| 0.15 | 5 | 0.5257485061 | $1.7 \times 10^{-7}$ | 0.5236 | $1.3 \times 10^{-2}$ |
| 0.15 | 15 | 0.5257681087 | $5.7 \times 10^{-15}$ | 0.5272 | $1.5 \times 10^{1}$ |
| 0.15 | 30 | 0.525768111101989 | $7.1 \times 10^{-21}$ | 0.7950 | $2.2 \times 10^{10}$ |
| 0.15 | 50 | 0.52576811110199309605 | $7.1 \times 10^{-21}$ | $5.5 \times 10^{4}$ | $5.1 \times 10^{31}$ |
| 0.20 | 5 | 0.5412382156 | $5.2 \times 10^{-6}$ | 0.5318 | $4.2 \times 10^{-1}$ |
| 0.20 | 20 | 0.5414194864 | $4.4 \times 10^{-13}$ | $-1.47$ | $1.1 \times 10^{9}$ |
| 0.20 | 40 | 0.54141951461230 | $6.9 \times 10^{-20}$ | $-7.3 \times 10^{6}$ | $9.7 \times 10^{37}$ |
| 0.20 | 80 | 0.54141951461272812011 | $6.9 \times 10^{-20}$ | $-6.5 \times 10^{25}$ | $1.9 \times 10^{120}$ |

validity of the HAM-based approach developed by Liao [18]. These two examples are widely used in textbooks of quantum mechanics. Unlike the perturbative approach [1-7], the HAM-based approach successfully provides the accurate convergent results for eigenvalues and eigenfunctions, as shown below.

The non-dimensional form of the time independent Schrödinger equation for a complex cubic anharmonic oscillator is given by [19]

$$H\psi_n(x) = \left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 + \mathrm{i}\lambda x^3\right)\psi_n(x) = E_n\psi_n(x),$$
$$n = 0, 1, \ldots, \text{ and } -\infty < x < \infty, \tag{15}$$

with an orthonormal basis

$$\psi_k^b(x) = \frac{1}{\pi^{1/4}(2^k\, k!)^{1/2}}\, \hat{H}_k(x)\, \exp\left(-\frac{x^2}{2}\right), \quad k = 0, 1, \ldots, N, \tag{16}$$

**Table 2** The convergent results of the eigenvalue $E_0$ of Eq. (15) given by the HAM-based approach for different values of $\lambda$ by means of the appropriate values of $c_0$ and $N$.

| $\lambda$ | $E_0$ | $c_0$ | $N$ |
|---|---|---|---|
| 0.01 | 0.5001373550 | $-1$ | 40 |
| 0.05 | 0.5033511914 | $-0.50$ | 40 |
| 0.1 | 0.5125381459 | $-0.33$ | 40 |
| 0.15 | 0.5257681111 | $-0.15$ | 45 |
| 0.2 | 0.5414195146 | $-0.10$ | 50 |
| 0.25 | 0.5583721242 | $-0.08$ | 50 |
| 0.3 | 0.5759252021 | $-0.06$ | 50 |
| 0.5 | 0.6458770810 | $-0.02$ | 55 |
| 0.75 | 0.7260238685 | $-0.005$ | 60 |
| 1 | 0.7973426075 | $-0.002$ | 60 |
| 1.25 | 0.8614046687 | $-0.002$ | 60 |
| 1.5 | 0.9196784830 | $-0.002$ | 60 |

$$E_k^b = k + \frac{1}{2}, \quad k = 0, 1, \ldots, N, \tag{17}$$

satisfying Eq. (2), where $\hat{H}_k(x)$ is the $k$th Hermite polynomial in $x$, $N$ is sufficiently large positive integer and

$$H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2. \tag{18}$$

Even in case of small disturbances, such as $\lambda = 0.15$ and 0.2, the residual error square of perturbation series increases quite significantly (see Table 1) and the perturbation series are divergent. It is found that the perturbation series is divergent for $\lambda \geq 0.1$, as shown in Fig. 1. However, by means of 6th-order HAM based iteration approach for $\lambda = 0.15$ and 0.2, the residual error square decreases sharply as the times of iteration increases, and convergent results of $E_0$ are obtained as 0.52576811110199309605 and 0.54141951461272812011, respectively, in accuracy of 20 digits. These HAM results agree with the results obtained by means of the homotopy-Padé technique [10, 12, 18]. The convergent HAM based results of $E_0$ are shown in Fig. 1 and Table 2. This shows the validity as well as superiority of the HAM-based approach over the perturbation approach for

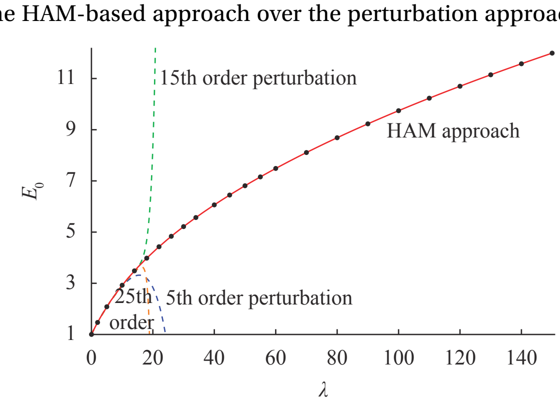


**Fig. 2.** Comparison of analytic approximations of the eigenvalue $E_0$ of Eq. (19) for different values of $\lambda$ when $a = 0$. Solid line: convergent results given by the HAM-based approach; solid circle: numerical results; long dashed line: the 5th-order perturbative result; dashed line: the 15th-order perturbative result; dashed dot dotted line: the 25th-order perturbative result.

the time independent Schrödinger equations.

Note that as the disturbance becomes larger, we should use a larger truncation number $N$ and a convergence-control parameter $c_0$ closer to zero. This has general meaning.

The non-dimensional time-independent Schrödinger equation for perturbed Pöschl-Teller potential is given by Ciftci et al. [8]

$$H\psi_n(x) = \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{a(a+1)}{\sin^2 x} + \lambda\cos^2 x\right]\psi_n(x) = E_n\psi_n(x),$$
$$n = 0, 1, \ldots \text{ and } 0 \le x \le \pi, \tag{19}$$

with an orthonormal basis

$$\psi_k^b(x) = \sqrt{\frac{k!\,(k+l)\,[\Gamma(l)]^2}{\pi\, 2^{(1-2l)}\,\Gamma(k+2l)}}\, C_k^l(\cos x)\sin^l x, \quad k = 0, 1, \ldots, N, \tag{20}$$

$$E_k^b = (k+l)^2, \quad k = 0, 1, \ldots, N, \tag{21}$$

**Table 3** The eigenvalue $E_0$ and residual error square of Eq. (19) in case of $a = 0$ and $\lambda = 18$ or 20 at different orders of approximation ($M$) given by the HAM-based approach using $N = 40$ and $c_0 = -0.5$ or $-0.47$, respectively, compared with perturbation results.

| $\lambda$ | $M$ | $E_0$ (HAM) | error (HAM) | $E_0$ (perturbation) | error (perturbation) |
|---|---|---|---|---|---|
| 18 | 10 | 3.9816208070 | $2.7 \times 10^{-7}$ | 3.983 | $7.6 \times 10^{-1}$ |
| 18 | 20 | 3.9814816093 | $1.5 \times 10^{-13}$ | 3.142 | 6.2 |
| 18 | 50 | 3.981481527702446 | $1.0 \times 10^{-31}$ | −29.80 | $2.9 \times 10^{5}$ |
| 18 | 100 | 3.98148152770244655541 | $7.7 \times 10^{-61}$ | $-3.4 \times 10^{5}$ | $4.6 \times 10^{19}$ |
| 20 | 5 | 4.2221579776 | $9.5 \times 10^{-4}$ | 3.49 | 2.8 |
| 20 | 20 | 4.2099195833 | $1.9 \times 10^{-12}$ | −0.99 | $4.2 \times 10^{2}$ |
| 20 | 50 | 4.20991940136222 | $3.9 \times 10^{-29}$ | $-3.7 \times 10^{3}$ | $5.5 \times 10^{13}$ |
| 20 | 100 | 4.20991940136222895618 | $1.4 \times 10^{-54}$ | $-3.5 \times 10^{9}$ | $3.7 \times 10^{35}$ |

**Table 4** The convergent results of eigenvalues $E_n$ ($n = 0, 1, 2$ and 3) of Eq. (19) at $a = 0$ for different values of $\lambda$ by means of the HAM-based approach using $N = 40$ and the appropriate values of $c_0$.

| $\lambda$ | $E_0$ | $E_1$ | $E_2$ | $E_3$ |
|---|---|---|---|---|
| 0.01 | 1.00249921899411516457 | 4.00499947916668079517 | 9.00500039038088420031 | 16.00500020833331900138 |
| 1 | 1.24242882598662974340 | 4.49479307863211894594 | 9.50366486704623913463 | 16.50208190103817298727 |
| 5 | 2.08298529320468367184 | 6.37066112500890732042 | 11.56933915693888693018 | 18.55120139840321147229 |
| 10 | 2.92366849417120532278 | 8.49247436673895637525 | 14.18570997013965417227 | 21.19483734691592869546 |
| 20 | 4.20991940136222895618 | 12.09946044548666536396 | 19.23632771369370141140 | 26.64821993716977690578 |
| 30 | 5.21474307442113992046 | 15.07251969901294834268 | 23.84564881697482335472 | 32.10850374889149650666 |
| 40 | 6.06344752074991164568 | 17.61784176404304428467 | 27.98606914468166043667 | 37.38138067862304300495 |
| 50 | 6.81116377466168925674 | 19.86700121109626252435 | 31.71794930664684257168 | 42.36733519814194107109 |
| 60 | 7.48699650257653336648 | 21.90065320110410358091 | 35.11661512331434779109 | 47.03734214335180575504 |
| 70 | 8.10835898359915907552 | 23.76993147941929539960 | 38.24731024045546251627 | 51.40299733081728707321 |
| 80 | 8.68661383308707315321 | 25.50893674401927591479 | 41.16070567916522741976 | 55.49397757698929289364 |
| 90 | 9.22965323133511648360 | 27.14153650448121471753 | 43.89500117963749620380 | 59.34443997266785447293 |
| 100 | 9.74322101531584071670 | 28.68513937775014914569 | 46.47905847337863056224 | 62.98648995274245978696 |
| 110 | 10.23165128933749180477 | 30.15288307322615558064 | 48.93500790252637350844 | 66.44790034914871858347 |
| 120 | 10.69830978643046275722 | 31.55496398422165438870 | 51.28013727513466046278 | 69.75186886811971957681 |
| 130 | 11.14587211465002765276 | 32.89948148670902614038 | 53.52820691765106416373 | 72.91754531708167755381 |
| 140 | 11.57650697021742532964 | 34.19299538242719427068 | 55.69036142948364037099 | 75.96073838144756045244 |
| 150 | 11.99200119612263392576 | 35.44090662990756309879 | 57.77576859706616873171 | 78.89457469325834971279 |

**Table 5** The convergent results of eigenvalues $E_n$ $(n = 0, 1, 2$ and $3)$ of Eq. (19) at $a$ = 0.5 for different values of $\lambda$ by means of the HAM-based approach using $N = 40$ and the appropriate values of $c_0$.

| $\lambda$ | $E_0$ | $E_1$ | $E_2$ | $E_3$ |
|---|---|---|---|---|
| 0.01 | 2.25199954297902658333 | 6.25428532557322463209 | 12.25466803024307397769 | 20.25480531381620911537 |
| 1 | 2.44554835541300395688 | 6.67469914377513472003 | 12.71791533039148730248 | 20.73169601305318091164 |
| 10 | 3.89389516880636419593 | 10.16620698489916480350 | 16.95629281567062397924 | 25.15631107380003495610 |
| 20 | 5.09414152740260350534 | 13.44338561468865863132 | 21.49663709190960663392 | 30.17091977566938310490 |
| 30 | 6.06362951752211741274 | 16.24087585650160522208 | 25.71066300676564320340 | 35.14728812065663659491 |
| 40 | 6.89384226882383707178 | 18.68943904688555403738 | 29.57764139776662527919 | 39.97540868634094803006 |
| 50 | 7.63017086323070120030 | 20.88084019195437405215 | 33.12805195325638730014 | 44.59388664039104362680 |
| 60 | 8.29820329564283579917 | 22.87698136560856723121 | 36.40605612593252098921 | 48.97929835147204614354 |
| 70 | 8.91382799523018675300 | 24.72006560902076615594 | 39.45433993918468776706 | 53.13231798154013344513 |
| 80 | 9.48764697411110775465 | 26.43969471838910190076 | 42.30928726676566424711 | 57.06657284447990498955 |
| 90 | 10.02713053251774216626 | 28.05730930465453244185 | 45.00045973314515870510 | 60.80117914461348443451 |
| 100 | 10.53776876739146817076 | 29.58891804161448147006 | 47.55151547659262516662 | 64.35639010512952539021 |
| 110 | 11.02373329542138675393 | 31.04680084615842519672 | 49.98139290480738611252 | 67.75140207925826343057 |
| 120 | 11.48828052205791492245 | 32.44060034510909407721 | 52.30536522061672134918 | 71.00345655360695673647 |
| 130 | 11.93400951180738726648 | 33.77804222517246892707 | 54.53587921100956465616 | 74.12763015868968544322 |
| 140 | 12.36303359661252832422 | 35.06542452520877615690 | 56.68319442813740575346 | 77.13694557705468701056 |
| 150 | 12.77709847357523410779 | 36.30795898004409138129 | 58.75586340205722377092 | 80.04260532216713975480 |

**Table 6** The convergent results of eigenvalues $E_n$ $(n = 0, 1, 2$ and $3)$ of Eq. (19) at $a$ = 2 for different values of $\lambda$ by means of the HAM-based approach using $N = 40$ and the appropriate values of $c_0$.

| $\lambda$ | $E_0$ | $E_1$ | $E_2$ | $E_3$ |
|---|---|---|---|---|
| 0.01 | 9.00124986330260921988 | 16.00299982501000008779 | 25.00374995813061262731 | 36.00414286130792181624 |
| 1 | 9.12365386959924016184 | 16.29826000777325693714 | 25.37456507849376416314 | 36.41431976372089656883 |
| 10 | 10.13187717670525692434 | 18.83499500360023909489 | 28.69431857380382659413 | 40.13941719090031751365 |
| 20 | 11.08425083698881516799 | 21.37874211769935584982 | 32.24040624850979415264 | 44.24321080409936506253 |
| 30 | 11.91432691518066124580 | 23.68408009040436704398 | 35.61203192009893769307 | 48.27469239115520320977 |
| 40 | 12.65629104026673191548 | 25.79480279979125937026 | 38.80618887068827779804 | 52.20760652946543395948 |
| 50 | 13.33174722263974848022 | 27.74570824844273458783 | 41.83109607064401944030 | 56.02574064216296777262 |
| 60 | 13.95508633660077936139 | 29.56399033259895926757 | 44.69974478059163010312 | 59.72102674537994466348 |
| 70 | 14.53634231949131640868 | 31.27086008637038891080 | 47.42650612348663801927 | 63.29136442152006178721 |
| 80 | 15.08278625079349909093 | 32.88296377147356359454 | 50.02543314504176105484 | 66.73869796967359298389 |
| 90 | 15.59986027045680047380 | 34.41349875526199809995 | 52.50949305166753183827 | 70.06750765539271396783 |
| 100 | 16.09174985076518123850 | 35.87305308878263409069 | 54.89029516626871650095 | 73.28370915170124297225 |
| 110 | 16.56174881675609514972 | 37.27022640324955805272 | 57.17806959362392525592 | 76.39389416164737602473 |
| 120 | 17.01250055641843471367 | 38.61208760741546477376 | 59.38176184528163563299 | 79.40483226693979835942 |
| 130 | 17.44616244630942662315 | 39.90451370622586760389 | 61.50917196478582591142 | 82.32316139592697182264 |
| 140 | 17.86452103843868204021 | 41.15244266856619709231 | 63.56710211325608157525 | 85.15520847621524630729 |
| 150 | 18.26907471873954711457 | 42.36006406667422857037 | 65.56149581319387130664 | 87.90689652399476512088 |

satisfying Eq. (2), where $C_k^l(\cos x)$ is the Gegenbauer polynomial, $N$ is a sufficiently large positive integer, $l = a + 1$ and

$$H_0 = -\frac{d^2}{dx^2} + \frac{a(a+1)}{\sin^2 x}. \quad (22)$$

The eigenvalue $E_n$ and its corresponding eigenfunctions $\psi_n(x)$ for different values of $a$ and $n = 0, 1, 2$ and 3 are evaluated using the HAM-based approach given by Liao [18] at small as well as very large disturbances. First, let us consider the eigenvalue $E_0$ at $a = 0$. It is very clear from Table 3 that for high disturbances such as $\lambda = 18$ and 20, the residual error square of perturbation series enlarges significantly as the order of approximation increases so that the perturbative results become invalid due to the divergence. The same is reported for higher disturbances. However, even with very high disturbances, the HAM-based approach successfully provides the convergent and accurate results for $E_0$ which agree well with the numerical results obtained by a constant reference potential perturbation method (CPM) [20] (see Fig. 2 and Table 4). The same is also noticed for $E_n$ and $\psi_n(x)$ with different values of $a$. The convergent results of ground state and excited energies $E_n$ ($n = 0, 1, 2$ and 3) in accuracy of 20 digits are listed in Tables 4, 5, and 6 for $a = 0, 0.5$ and 2, respectively. These results also agree well with its homotopy-Padé approximation [10, 12, 18]. This further indicates the general validity of HAM-based approach for the Schrödinger equation in quantum mechanics, and its superiority over the traditional perturbative approaches.

In this article, two examples, i.e. the complex cubic anharmonic oscillator and the perturbed Pöschl-Teller potential, are successfully solved by means of the HAM-based approach first proposed by Liao [18] for the time-independent Schrödinger equations in quantum mechanics. Note that perturbation approach becomes invalid when the disturbance enlarges. However, convergent results of eigenvalue and eigenfunction are always obtained by means of the HAM-based approach, even when the disturbances are quite large, which can be directly compared with the experiment data to improve the accuracy of the experimental findings and/or physical theories. This further illustrates the general validity of the HAM-based approach proposed in Ref. [18] for the time-independent Schrödinger equations in quantum mechanics, and its superiority over the traditional perturbative approaches. This study suggests that the HAM might provide us a new, powerful alternative to solve some complicated problems in quantum mechanics, including many-body problems which we will consider in future.

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