Chaos

# Interplay between random fluctuations and rate dependent phenomena at slow passage to limit-cycle oscillations in a bistable thermoacoustic system

Cite as: Chaos 29, 031102 (2019); https://doi.org/10.1063/1.5088943
Submitted: 10 January 2019 . Accepted: 11 March 2019 . Published Online: 28 March 2019

Vishnu R. Unni, E. A. Gopalakrishnan, K. S. Syamkumar, R. I. Sujith, Elena Surovyatkina, and Jürgen Kurths

# Interplay between random fluctuations and rate dependent phenomena at slow passage to limit-cycle oscillations in a bistable thermoacoustic system

Cite as: Chaos **29**, 031102 (2019); doi: 10.1063/1.5088943
Submitted: 10 January 2019 · Accepted: 11 March 2019 ·
Published Online: 28 March 2019



Vishnu R. Unni,[1,a)] E. A. Gopalakrishnan,[2] K. S. Syamkumar,[3] R. I. Sujith,[3] Elena Surovyatkina,[4,5,6] and Jürgen Kurths[4]

## AFFILIATIONS

[1] Department of Mechanical and Aerospace Engineering, University of California San Diego, San Diego, California 92093, USA
[2] Center for Computational Engineering and Networking, Amrita School of Engineering, Coimbatore 641112, India
[3] Indian Institute of Technology Madras, Chennai 600036, India
[4] Potsdam Institute for Climate Impact Research, Berlin 14412, Germany
[5] Space Dynamics and Data Analysis Department, Space Research Institute of Russian Academy of Sciences, Moscow 117997, Russian Federation
[6] Kamchatka State Technical University, Petropavlovsk-Kamchatsky 683003, Russian Federation

[a)] **Electronic mail:** vunni@ucsd.edu

## ABSTRACT

We study the impact of noise on the rate dependent transitions in a noisy bistable oscillator using a thermoacoustic system as an example. As the parameter—the heater power—is increased in a quasi-steady manner, beyond a critical value, the thermoacoustic system undergoes a subcritical Hopf bifurcation and exhibits periodic oscillations. We observe that the transition to this oscillatory state is often delayed when the control parameter is varied as a function of time. However, the presence of inherent noise in the system introduces high variability in the characteristics of this critical transition. As a result, if the value of the system variable—the acoustic pressure—approaches the noise floor before the system crosses the unstable manifold, the effect of rate on the critical transition becomes irrelevant in determining the transition characteristics, and the system undergoes a noise-induced tipping to limit-cycle oscillations. The presence of noise-induced tipping makes it difficult to identify the stability regimes in such systems by using stability maps for the corresponding deterministic system.



**Noise is an inherent part of practical systems. When a system is nonlinear, noise can have nontrivial effects on its dynamics. We study the effect of inherent noise on dynamic bifurcations in a nonlinear system as a parameter of the system is varied in time. We show that noise can have varied effects on the dynamic bifurcation in a system, depending on the initial conditions of the system and the rate at which the parameter of the system is varied. We use a Rijke tube to experimentally demonstrate our findings from the theoretical analysis.**

## I. INTRODUCTION

Many physical and engineering systems exhibit a sudden qualitative change in their dynamics to an infinitesimal change in any of the system parameters.[1] Often, one finds that these transitions, also known as bifurcations, are detrimental and affect the performance of the system. The analysis of bifurcations in a system of interest is performed, either through experiments or numerical simulations, to characterize the different regimes pertaining to various dynamical states of the system. In bifurcation analysis, the system is generally considered to be autonomous. In other words, the time dependent nature of the system parameters are often neglected. Such static bifurcation studies are conducted by identifying the asymptotic state of the system for different values of control parameters. Furthermore, different initial conditions are used to explore the possibility of multiple stable states of the system. Many real systems are non-autonomous in nature,[2,3] and the dynamic transitions in such systems are a function of the rate of change of system's parameters.[4] We can find numerous studies on rate dependent bifurcations using canonical models of standard bifurcations.[5,6] Apart from numerical investigations, there is a limited number of studies on rate dependent bifurcations by performing physical experiments.[7–12] One major difference that can be found in rate dependent transitions as compared to quasi-static transitions is the rate-induced delay in the transition. This delay in the transition is attributed to the fact that as a result of the temporal variation of the control parameter, the system continues to hover around the unstable dynamical state for a finite amount of time even after the system loses its stability.[4] In this paper, we explore the effect of inherent fluctuations in a system on the rate-induced delay observed in the dynamic transitions of the system.

The effect of stochastic fluctuations in the dynamic characteristics of non-autonomous systems is of immense practical interest. The rate-induced delay observed in rate-dependent transitions is found to be affected or even destroyed in the presence of fluctuations.[13] Here, fluctuations in a system could be the result of the noise present inside or around the system. Alternatively, it could be the result of the projection of dynamics of the system in its higher dimensions onto the dimension along which the bifurcation is occurring. Furthermore, the delay observed in transitions depends upon the rate of change of the control parameter and the initial condition of the dynamical system.[14] This dependency of delay on fluctuations, rate of change of the control parameter, and the initial conditions makes it difficult to adopt the stability map constructed using a quasi-static analysis to understand the stability margins of a system. In many practical systems, an infinitesimal change in any of the system parameters leads to catastrophic transitions and a perfect understanding of stability regimes is essential for ensuring their proper performance.

One such physical system where an infinitesimal change in the system parameter leads to a catastrophic transition is a combustor system where heat energy is used for producing mechanical or electrical power. Gas turbine engines and rocket engines are examples of combustor systems. In these systems, a heat source such as a flame is located in a confinement. One major impediment observed in the operation of these systems is the sudden onset of large amplitude pressure oscillations which affect the performance of the system and at times even lead to the failure of the system. The main reason for the onset of large amplitude pressure oscillations is a positive coupling between the inherent acoustic fluctuations and the unsteady heat release rate fluctuations. As the above phenomenon is the result of the interaction between the acoustic fluctuations and heat release rate fluctuations, it is known as thermoacoustic instability and the associated system as a thermoacoustic system.[15]

In all industrial thermoacoustic systems, the control parameter is varied as a function of time in order to meet the varying demands of power generation. Hence, it is highly pertinent to investigate the rate dependent transitions in a thermoacoustic system. A recent study on the effect of rate dependent transitions in thermoacoustic systems was performed by Tony *et al.*[8] They found that a bistable thermoacoustic system will undergo a rate-induced tipping (R-tipping) when the rate of change of the control parameter is above a critical threshold for a specific set of initial conditions. Here, tipping refers to the state point crossing the unstable manifold. Recently, Bonciolini *et al.*[16] conducted an experimental study to analyze the rate dependent transitions in a turbulent combustor. They also performed numerical experiments on a surrogate oscillator model and reported a rate-induced delay in the thermoacoustic system depicting a subcritical Hopf bifurcation in the presence of noise.

However, these studies[8,16] do not consider in detail the effect of initial conditions of the state variable and the control parameter. The systems considered in the aforementioned studies are bistable systems, and it is important to understand the influence of the initial conditions on characteristics of the dynamic bifurcation in such systems. Moreover, as most physical systems work in the presence of fluctuations, it is highly essential to investigate the effect of inherent fluctuations on the dynamics of rate dependent transitions. In this study, we explore the effect of inherent fluctuations and also the effect of initial conditions of the state variable on the rate-dependent transitions in a thermoacoustic system, both by performing physical and numerical experiments.

We consider a thermoacoutic system which exhibits a transition from a non-oscillatory to oscillatory state via a subcritical Hopf bifurcation as we change the system parameters. Here, we use a thermoacoustic system known as a Rijke tube, which behaves as a bistable oscillator under a range of operating conditions.[17] A Rijke tube consists of a duct with a heat source located inside. A flow of air is established inside the duct either through buoyancy driven convection (for a vertical Rijke tube) or forced convection (for a horizontal Rijke tube). For the current study, we use a horizontal Rijke tube. The positive feedback between the inherent acoustic fluctuations of the duct and the unsteady heat release rate of the heat source can cause large amplitude pressure oscillations inside the Rijke tube as a result of thermoacoustic instability.[18–22] However, all these studies considered Rijke tube as an autonomous system, and the effect of temporal variation of control parameter was not taken into account. In this paper, we will explore the non-autonomous behavior of a horizontal Rijke tube.

Section II details the experimental setup, and Sec. III describes the model. Results and discussions are given in Sec. IV, and the conclusions are detailed in Sec. V.

## II. EXPERIMENTAL SETUP

The Rijke tube used for the present study consists of a horizontal duct of square cross-section with a heat source—an electrically heated wire mesh—located within the duct. When air flows through the duct, an acoustic standing wave is established in the Rijke tube if the heater power is above a critical value. The electrical power used to heat the wire mesh is provided using a DC power supply (TDK-Lambda, GEN 8-400, 0-8 V, 0-400 A). The air flow rate is maintained with the help of a mass flow controller (Alicat Scientific, MCR Series, 100 *SLPM*); uncertainty is $\pm$(0.8% of reading + 0.2% of full scale). A piezoelectric sensor (PCB103B02 with a sensitivity of 223.4 mV/kPa

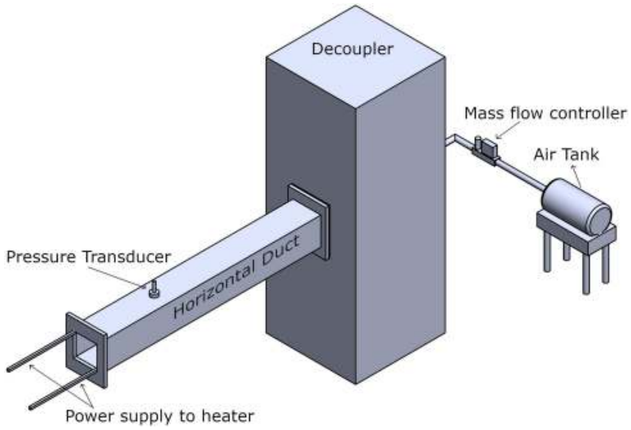


**FIG. 1.** Schematic of the horizontal Rijke tube used for the experimental study. Air flow is introduced into the Rijke tube via a decoupler which ensures that the inlet fluctuations do not affect the dynamics of the Rijke tube. The other end of the Rijke tube is open to the atmosphere. This design ensures that both ends of the Rijke tube remain acoustically open (i.e., the acoustic pressure fluctuation $p'$ at the boundary is zero).

and an uncertainty of $\pm 0.15$ Pa) is used to measure the acoustic pressure within the Rijke tube, and the temperature is measured with the help of a K-type thermocouple (uncertaininty $\pm 0.1$ K). A detailed schematic of the experimental setup is shown in Fig. 1.

The experiments for quasi-static bifurcation was conducted by varying the parameter (heater power) in a quasi-steady manner. That is, for each small change in the value of the parameter, the system was allowed to evolve for 120 s before recording the asymptotic state of the system corresponding to that parameter.The rate of change between each step of the quasi-static bifurcation study is 50 W/s. However, since the change in heater power in each step is small and is $\approx 10$ W, and a 120 s evolution time is provided for the transients to settle, the average rate of change of power is $\approx 0.1$ W/s. The dynamic bifurcation (parameter varying at a specific rate) was studied by varying the heater power as a linear function of time.

## III. THE MODEL

We study the dynamic bifurcations in a Rijke tube by analyzing the characteristics of dynamic bifurcation in a basic system of differential equations in polar coordinates [Eqs. (1) and (2)] that exhibit Hopf bifurcation when the control parameter, $\mu$, is varied in time. Previously, Subramanian *et al.*[23] and Etikyala and Sujith[17] showed that such a system of equations derived from the Stuart-Landau model can be used to study the dynamics of a Rijke tube close to bifurcation. We have introduced additive noise, $\eta$, in the model to mimic the inherent fluctuations present in a Rijke tube:

$$\dot{x} = \mu x + a x^3 - b x^5 + \eta, \tag{1}$$

$$\dot{\theta} = 1. \tag{2}$$

Here, $x$ is the variable representing the amplitude of oscillations in the Rijke tube. $\dot{\theta}$ is the frequency of oscillations, which is assumed to be a constant in this case. $\mu$ represents the linear decay rate of the amplitude. In the present case, $a = 1$ and $b = 0.1$ were selected arbitrarily, and it does not affect the conclusions drawn from the study. $\eta$ is the noise present in the system, which in this case is Gaussian white noise of zero mean and variance 0.001.

For the analysis, we focus on the effect of noise on the variability in the characteristics of the dynamic bifurcation during a slow passage across the Hopf point. For simplicity, we assume that only the fundamental mode of the Rijke tube gets excited as a result of thermoacoustic instability. This is a good approximation for the present system as we observe that only the fundamental mode is excited in the experiments.

Note that by definition, $x$ is the amplitude of oscillations and is always greater than or equal to zero. However, the noise $\eta$ added in the system can cause it to become negative at some time instants. In order to circumvent this issue, first we convert the system of Eqs. (1) and (2) from cylindrical coordinates $(x, \theta)$ to Cartesian coordinates $(p, q)$ using the transformation $p = x\cos(\theta)$ and $q = x\sin(\theta)$. Equations (3) and (4) represent the new system of equations:

$$\dot{p} = p\left[\mu + a(p^2+q^2) - b(p^2+q^2)^2 + \frac{\eta}{\sqrt{p^2+q^2}}\right] - q, \tag{3}$$

$$\dot{q} = q\left[\mu + a(p^2+q^2) - b(p^2+q^2)^2 + \frac{\eta}{\sqrt{p^2+q^2}}\right] - p. \tag{4}$$

This stochastic differential equation is then solved using the Euler-Maruyama method. After obtaining the solution in terms of $(p, q)$, the temporal evolution of $x$ is calculated by using the transformation, $x = \sqrt{p^2 + q^2}$. This ensures that $x$ always remains greater than or equal to zero. Furthermore, note that the terms that contain noise in Eqs. (3) and (4) are singular for $p = q = 0$. However, we observe that this is not an issue for numerically solving the differential equations. Due to the presence of noise in the system and the computational noise, the system almost never reaches the state $p = q = 0$. Furthermore, we ensure that the initial condition of the system is never $p = q = 0$. Throughout this paper, the initial condition in $\theta$ is always considered to be equal to zero and initial condition in $x$ is always greater than zero. The results from the analysis of dynamic bifurcations in the numerical model are discussed in Sec. IV.

## IV. RESULTS

The behavior of a system undergoing a Hopf bifurcation, described by Eqs. (1) and (2), is depicted in Fig. 2. Dashed and continuous black lines represent unstable and stable limit cycle for $x > 0$. At $x = 0$, they represent unstable and stable fixed points. Collectively, we refer to them as unstable and stable manifolds of the system in the absence of noise. For $\mu < 0$, the system has a stable fixed point at $x = 0$. For $\mu > 0$, the fixed point becomes unstable. In the forward transition, i.e., quasi-steady variation of $\mu$ from negative to positive value, the appearance of limit cycle oscillations is abrupt. The parameter at which the bifurcation happens, $\mu = 0$, is called the Hopf point or critical point. In backward transition, i.e., quasi-steady variation of $\mu$ from positive to negative value, the system undergoes a fold bifurcation at $\mu = -2$. We can see from Fig. 2(a) that the system can either remain in the state of a stable fixed point or in the state of stable limit cycle oscillations for $\mu$ ranging from $-2$ to 0. Thus, there exists a bistable region from $\mu = -2$ to $\mu = 0$.

Until, we examined the static bifurcation for the bi-stable oscillator. However, in almost all real systems, the parameters vary as a function of time which necessitates the study of dynamic bifurcations. In order to perform a dynamic bifurcation analysis, we vary the control parameter $\mu$ as

$$\mu = \mu_0 + \dot{\mu} t. \tag{5}$$

In this study, we consider only linear variation of parameter in time and hence $\dot{\mu}$ is constant in time. We perform a dynamic bifurcation analysis by solving Eqs. (1) and (2) where the control parameter $\mu$ varies as per Eq. (5).

In Fig. 2(a), we look at a case where the initial value $x$, i.e., $x_0$, is non zero and is equal to 2. This implies that in the phase space corresponding to the system, the initial state point is a finite distance away from the fixed point. The initial value of the parameter, $\mu_0$, is chosen to be $-5$. As time evolves, the system parameter $\mu$ is varied at a fixed rate. In Fig. 2(a), we show the dynamic bifurcation for three different values of $\dot{\mu}$ represented by the green, red, and blue trajectories. In all three cases, initially, the system moves towards the fixed point at $x = 0$ when the control parameter is sufficiently far away from the critical value of the control parameter (i.e., $\mu = 0$). As we approach the Hopf point, the system crosses the unstable manifold at a parameter which is found to depend upon the value of $\dot{\mu}$. When starting from the same $\mu_0$, for higher value of $\dot{\mu}$, the system reaches the unstable manifold in lesser time compared to that for a lower value of $\dot{\mu}$. This implies that for the higher value of $\dot{\mu}$, the stable manifold attracts the system for shorter duration compared to the case where the rate of change of the parameter is low. Hence, for the same initial conditions in the state point, when $\dot{\mu}$ is higher, the system crosses the unstable manifold at a higher value of $x$ compared to a lower value of $\dot{\mu}$. This further implies that as the system crosses the unstable manifold, it starts to get attracted by the stable limit cycle branch at a lower value of $\mu$ for a higher value of $\dot{\mu}$.

This rate dependency in the loss of stability of the system causes the dynamic bifurcation to have the particular trend as seen in Fig. 2(a). The trajectory starts to approach the limit-cycle at lower parameter values for higher values of $\dot{\mu}$. However, it should be noted that the different trajectories intersect as they approach the limit cycle branch. For a different initial condition, there can be a scenario where the trajectories intersect much before they reach the stable limit cycle branch (not shown here). The observations from Fig. 2(a) suggest that the temporal variation of $\mu$ introduces an R-tipping in the bistable oscillator. For higher rates, the system tips at a lower parameter value. However, in this case, the initial condition of the system is at a finite distance away from the fixed point.

Now, we will consider a case where the initial condition of the system is at/very close to the fixed point $x = 0$, and the dynamics of the system is highly affected by the noise in the system. In Fig. 2(b), we consider the dynamic bifurcation at the same $\dot{\mu}$ values as in Fig. 2(a). However, here $x_0 = 0.0001$, i.e., very close to the fixed point. In a system without noise, if the initial condition is $x_0 = 0$, when $\mu$ is varied, the system does not exhibit oscillatory behavior for any value of $\mu$, since the system continues to remain in the fixed point even though the stability of the fixed point changes for $\mu > 0$. However, in a noisy system, even though the initial condition is $x_0 = 0$, the system can

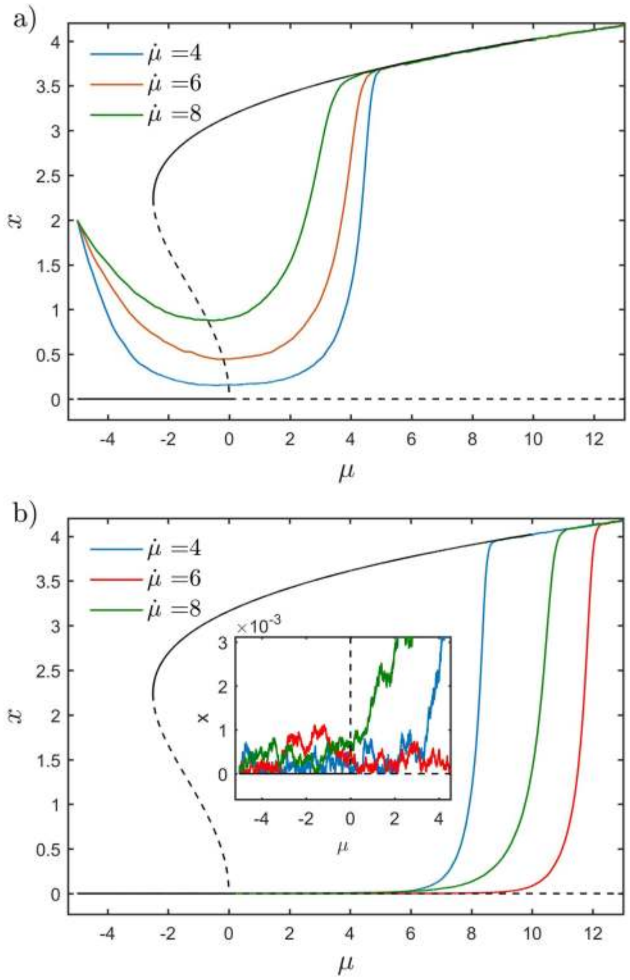


**FIG. 2.** The effect of rate during slow passage through the critical point in a system with noise that undergoes a sub-critical Hopf bifurcation: (a) the passage at three different rates, initial condition: $x_0 = 2$, $\mu_0 = -5$ and (b) the passage at three different rates, initial condition: $x_0 = 0.0001$, $\mu_0 = -5$. The rate dependency is lost when the system crosses the unstable manifold very close to stable fixed points.

exhibit limit cycle oscillations for $\mu > 0$, since the noise perturbs the system from the unstable fixed point causing it to drift towards the stable limit cycle. In Fig. 2(b), we can see that the approach to the limit cycle for trajectories of different rates does not follow a specific trend. That is, the order in which trajectories approach the limit cycle does not depend on their respective $\dot{\mu}$.

In order to understand the reason for this, we zoom-in close to the Hopf point and see how the different trajectories cross the unstable manifold. The zoomed-in view is given in the inset of Fig. 2(b). We find that noise determines the manner in which the system crosses the unstable manifold. Thus, unlike in the case described in Fig. 2(a), the order in which the different trajectories cross the unstable manifold is not dependent on their corresponding $\dot{\mu}$ and is rather dependent on the individual noise realization for the trajectories.

Furthermore, one can note that even if one trajectory crosses the unstable manifold before another, the specific realization of noise can still trap that trajectory close to the unstable fixed point delaying the growth of oscillations in the system. Note that the trajectory corresponding to $\dot{\mu} = 6$ (red) crosses the unstable manifold before the trajectory corresponding to $\dot{\mu} = 4$ (blue). This can be inferred by observing that the trajectory corresponding to $\dot{\mu} = 6$ (red) crosses the unstable manifold at a higher value of $x$ compared to that for the trajectory corresponding to $\dot{\mu} = 4$ (blue). Among the two trajectories, if the noise was not present, the trajectory that crosses the unstable manifold at a higher value of $x$ is the one that will move towards the stable manifold at a lower value $\mu$. However, in the presence of noise, this trend is lost. Here, we observe that the trajectory corresponding to $\dot{\mu} = 4$ (blue) starts approaching the stable limit cycle oscillations at a lower value of $\mu$ compared to that of the trajectory corresponding to $\dot{\mu} = 6$ (red), even though the red trajectory crosses the unstable manifold at a slightly higher value of $x$.

In order to study the variability of the trajectories due to noise, we explore the transition characteristics for multiple realizations of the dynamic bifurcation (Fig. 3). We consider two conditions: $\dot{\mu} = 6$, $x_0 = 2$, $\mu_0 = -5$ (gray curves) and $\dot{\mu} = 6$, $x_0 = 0.0001$, $\mu_0 = -5$ (blue curves). The difference here is in the initial value of the variable $x$. For each condition, we plotted 100 realizations of transitions in the same graph. We can see that for both cases, due to noise, there is a considerable variability in the trajectories along which each realization approaches the limit cycle oscillations. However, it is very clear that the variability is higher for the case for which $x_0 = 0.0001$. This is due to the fact that the influence of noise in determining the transition characteristics is significant when the trajectory passes very close to the $x = 0.001$ line, where the fluctuations introduced due to noise are comparable to the value of $x$.

Furthermore, we investigate effects of the rate in the dynamic bifurcation using experiments. We initially perform quasi-static bifurcation analysis to understand the stability regimes associated with the system. The asymptotic value of the acoustic pressure is noted for each value of the heater power ($K$). The bifurcation diagram shown in Fig. 4 (blue dots—forward bifurcation as $K$ is increased and pink dots—reverse bifurcation as $K$ is reduced) represents the variation of the acoustic pressure with heater power. We clearly observe that the transition points to and from the oscillatory state differ for the forward and backward bifurcation, respectively, which leads to the presence of a bistable zone. The sudden increase in the acoustic pressure amplitude along with the presence of the bistable zone indicates that the transition observed is a subcritical Hopf bifurcation.

Furthermore, we proceed to understand the effect of variation of the control parameter as a function of time on the bifurcation in the system. For this purpose, we conduct an experiment by varying the heater power as a linear function of time with different rates. We observe that there is a delay in the bifurcation (i.e., the system reaches the limit cycle oscillation at a higher value of $K$) when the parameter is varied in time as opposed to quasi-static manner. However, similar to the observation from the numerical simulations, for individual realizations, there is no clear trend observed in this delay vs. the rate of change of the control parameter, $\dot{K}$ (Fig. 4) as inferred from the overlapping of trajectories of different rates each represented by a different color.

However, interestingly, when we consider an average trajectory corresponding to each condition, the rate dependent behavior is preserved. The average trajectory for a given condition is obtained by ensemble averaging $K$ for a given value of $x$ across various realizations

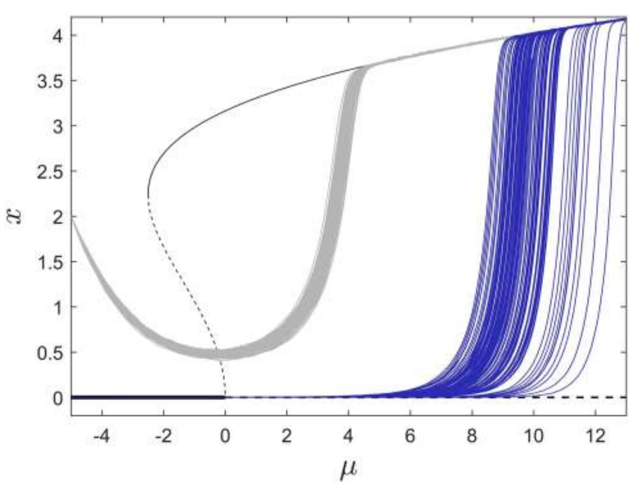


**FIG. 3.** The transition characteristics for multiple realizations (100) of dynamic bifurcation at the same rate of change of control parameter. We consider two conditions: $\dot{\mu} = 6$, $x_0 = 2$, $\mu_0 = -5$ (gray curves) and $\dot{\mu} = 6$, $x_0 = 0.0001$, $\mu_0 = -5$ (blue curves).

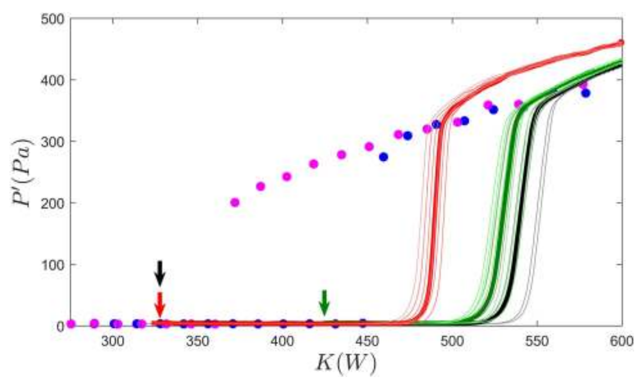


**FIG. 4.** The forward (blue dots) and reverse path (magenta dots) of quasi-static bifurcation for the Rijke tube is represented in the figure. The different realizations of dynamic bifurcations when the parameter ($K$, heater power) is varied as a function of time are represented by the continuous thin lines. The bold lines represent the average trajectory for multiple realizations with the same initial condition and the rate of change of parameter. Red line: $K_0 = 325$ W, $\dot{K} = 12.5$ W/s; black line: $K_0 = 325$ W, $\dot{K} = 5$ W/s; and green line: $K_0 = 425$ W, $\dot{K} = 5$ W/s. The arrows of different colors indicate the initial conditions in $K$ (i.e., $K_0$) for trajectories of corresponding color.

for the same conditions. The bold colored lines in Fig. 4 represent average trajectories corresponding to the individual trajectories of corresponding colors. We find that for the same value of $K_0$, the average trajectory corresponding to a lower $\dot{K}$ approaches the limit cycle at a lower value of $K$. Also, for the same value of $\dot{K}$, the average trajectory corresponding to lower $K_0$ approaches the limit cycle at a higher value of $K$. We observe from the numerical study that when the system is noisy and the initial amplitude of the oscillation is comparable with the noise in the system, then the critical transition in the system as the result of the dynamic bifurcation is mostly dependent upon the noise in the system but does not show the characteristics of a rate-dependent tipping. Tony *et al.*[8] discovered preconditioned rate tipping in a thermoacoustic system. They argued that there will always be a rate below which the R-tipping will not be present. Our observation shows that this indeed is true in the case of a noisy bi-stable oscillator, and we further show that below a critical rate, the system undergoes a Noise induced tipping (N-tipping) instead of a R-tipping. N-tipping happens when tipping is controlled by the noise in the system. We also demonstrate that while individual realizations undergo a N-tipping, when we consider an average trajectory, the effect of rate is preserved. We would also like to note that, for every initial condition, the tipping is influenced by the characteristics of the noise $\eta$ and the rate of change of the control parameter ($\dot{\mu}$). For a given initial condition ($x_1$), for two different values of $\dot{\mu}$ such that $\dot{\mu}_1 > \dot{\mu}_2 > \dot{\mu}_{critical}$, the relative difference in $\mu$ at which the trajectories cross the unstable manifold, $\delta\mu$, has a probability distribution. The nature of this distribution is depended on the initial condition in $x$, $\mu$, $\dot{\mu}_1 - \dot{\mu}_2$, $\dot{\mu}_1$, $\dot{\mu}_{critical}$, and the topology of the unstable manifold. Here, $\dot{\mu}_{critical}$ is the critical rate of change of parameter for which the system approaches the noise floor before crossing the unstable manifold. A detailed analysis of the tipping characteristics will be performed in a future study.

Majumdar *et al.*[3] showed that for a transition through a transcritical bifurcation point, the delay of exchange of stability is independent of the rate of change of control parameter and proportional to the initial value of the dynamical variable. In contrast, Bonciolini *et al.*[16] suggested that at a higher rate of the control parameter, there is an increased delay in the onset of thermoacoustic instability through subcritical Hopf bifurcation. While this is true when we consider the average delay in the onset of thermoacoustic instability as shown by Bonciolini *et al.*,[16] we uncover that the inherent fluctuations in a physical system can induce high variability in the transition to the oscillatory state for individual realizations. In such a scenario, we argue that constructing stability diagrams for such systems is difficult, as the noise smears the stability boundaries. Thus, for such systems, there is a need to develop early warning signals to predict transitions for each realization of a critical transition.

## V. CONCLUSION

We have shown that the rate dependency of the onset of oscillations in a bi-stable oscillator undergoing a sub-critical Hopf bifurcation is highly influenced by the initial condition and the noise in the system. If the system crosses the unstable manifold close to the stable fixed point such that the noise level is comparable to the amplitude of the system variable when the system crosses the unstable manifold, the system undergoes an N-tipping and the effect of rate in determining the characteristics of the critical transition is diminished. Using experiments, we show that this indeed is the case in the dynamic bifurcations observed in a horizontal Rijke tube. While on the average, systems exhibit a rate dependency as predicted by the models that do not include the inherent fluctuations in the system, for individual realizations, there is less predictability of the stability margins. Thus, we argue that in practical systems that exhibit such dynamic bifurcations, the variability in dynamic bifurcation as the result of the effect of noise warrants the need for early warning signals that forewarns the impending critical transition.

## ACKNOWLEDGMENTS

V.R.U., E.A.G., K.S.S., and R.I.S. thank DST, India, and DAAD, Germany, for funding the Indo-German joint project (F. No. DST/INT/FRG/DAAD/P-12/2017). J.K. thanks DAAD (Project No. 57318077), and E.S. acknowledge financial support from the Russian Foundation for Basic Research (RFBR) (No. 16-07-01186).

## REFERENCES


1. S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (CRC Press, 2018).
2. J. R. Tredicce, G. L. Lippi, P. Mandel, B. Charasse, A. Chevalier, and B. Picqué, "Critical slowing down at a bifurcation," Am. J. Phys. **72**, 799–809 (2004).
3. A. Majumdar, J. Ockendon, P. Howell, and E. Surovyatkina, "Transitions through critical temperatures in nematic liquid crystals," Phys. Rev. E **88**, 022501 (2013).
4. S. M. Baer, T. Erneux, and J. Rinzel, "The slow passage through a Hopf bifurcation: Delay, memory effects, and resonance," SIAM J. Appl. Math. **49**, 55–71 (1989).
5. L. Bilinsky and S. Baer, "Slow passage through a Hopf bifurcation in excitable nerve cables: Spatial delays and spatial memory effects," Bull. Math. Biol. **80**, 130–150 (2018).
6. P. Ashwin, C. Perryman, and S. Wieczorek, "Parameter shifts for nonautonomous systems in low dimension: Bifurcation-and rate-induced tipping," Nonlinearity **30**, 2185 (2017).
7. W. Scharpf, M. Squicciarini, D. Bromley, C. Green, J. Tredicce, and L. Narducci, "Experimental observation of a delayed bifurcation at the threshold of an argon laser," Opt. Commun. **63**, 344–348 (1987).
8. J. Tony, S. Subarna, K. Syamkumar, G. Sudha, S. Akshay, E. Gopalakrishnan, E. Surovyatkina, and R. Sujith, "Experimental investigation on preconditioned rate induced tipping in a thermoacoustic system," Sci. Rep. **7**, 5414 (2017).
9. D. Premraj, K. Suresh, T. Banerjee, and K. Thamilmaran, "An experimental study of slow passage through Hopf and pitchfork bifurcations in a parametrically driven nonlinear oscillator," Commun. Nonlinear Sci. Numer. Simul. **37**, 212–221 (2016).
10. A. Pisarchik, "Dynamical tracking of unstable periodic orbits," Phys. Lett. A **242**, 152–162 (1998).
11. A. Pisarchik and R. Corbalán, "Shift of attractor boundaries in a system with a slow harmonic parameter perturbation," Phys. D Nonlinear Phenom. **150**, 14–24 (2001).
12. A. Pisarchik, R. Jaimes-Reátegui, C. A. Magallón-García, and C. O. Castillo-Morales, "Critical slowing down and noise-induced intermittency in bistable perception: Bifurcation analysis," Biol. Cybern. **108**, 397–404 (2014).
13. N. Berglund and B. Gentz, "Pathwise description of dynamic pitchfork bifurcations with additive noise," Probab. Theory Relat. Fields **122**, 341–388 (2002).
14. N. Berglund and B. Gentz, "On the noise-induced passage through an unstable periodic orbit II: General case," SIAM J. Math. Anal. **46**, 310–352 (2014).
15. M. P. Juniper and R. Sujith, "Sensitivity and nonlinearity of thermoacoustic oscillations," Annu. Rev. Fluid Mech. **50**, 661–689 (2018).
16. G. Bonciolini, D. Ebi, E. Boujo, and N. Noiray, "Experiments and modelling of rate-dependent transition delay in a stochastic subcritical bifurcation," R. Soc. Open Sci. **5**, 172078 (2018).

[17] S. Etikyala and R. Sujith, "Change of criticality in a prototypical thermoacoustic system," Chaos **27**, 023106 (2017).
[18] K. I. Matveev, "Thermoacoustic instabilities in the Rijke tube: Experiments and modeling," Ph.D. thesis (California Institute of Technology, 2003).
[19] M. P. Juniper, "Triggering in the horizontal rijke tube: Non-normality, transient growth and bypass transition," J. Fluid Mech. **667**, 272–308 (2011).
[20] P. Subramanian, "Dynamical systems approach to the investigation of thermoacoustic instabilities," Ph.D. thesis (Indian Institute of Technology Madras, 2011).
[21] S. Mariappan, "Theoretical and experimental investigation of the non-normal nature of thermoacoustic interactions," Ph.D. thesis (Indian Institute of Technology Madras, 2011).
[22] E. Gopalakrishnan and R. Sujith, "Effect of external noise on the hysteresis characteristics of a thermoacoustic system," J. Fluid Mech. **776**, 334–353 (2015).
[23] P. Subramanian, R. Sujith, and P. Wahi, "Subcritical bifurcation and bistability in thermoacoustic systems," J. Fluid Mech. **715**, 210–238 (2013).