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G. Ramesh, B. Sudip Ranjan, and D. Venku Naidu

Cyclic Composition operators on Segal-Bargmann space

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Abstract: We study the cyclic, supercyclic and hypercyclic properties of a composition operator C_ϕ on the Segal-Bargmann space $\mathcal{H}(\mathcal{E})$, where $\phi(z) = Az + b$, A is a bounded linear operator on $\mathcal{E}$, $b \in \mathcal{E}$ with $\|A\| \leqslant 1$ and A^*b belongs to the range of $(I - A^*A)^{\frac{1}{2}}$. Specifically, under some conditions on the symbol ϕ we show that if C_ϕ is cyclic then A^* is cyclic but the converse need not be true. We also show that if C_ϕ^* is cyclic then A is cyclic. Further we show that there is no supercyclic composition operator on the space $\mathcal{H}(\mathcal{E})$ for certain class of symbols ϕ.

Keywords: Segal-Bargmann space, composition operator, Cyclic operator, reproducing kernel Hilbert space, hypercyclic and supercyclic operators

MSC: Primary 47B33; Secondary 46E22, 47A16

1 Introduction

It is known that every bounded linear operator on an infinite dimensional complex separable Hilbert space is the sum of two hypercyclic operators [3, p. 50]. It is interesting to note that this result holds true with the summands being cyclic operators. Therefore, it is very important to study the cyclic operators in order to study bounded operators. We know that every hypercyclic operator is cyclic and supercyclicity is a property which is intermediate between these two.

Since the closed linear span of $\mathrm{Orb}(T, x)$ is the smallest closed T-invariant subspace that contains the vector x, the cyclic property is connected with the study of invariant subspaces. Analogously, hypercyclicity has the same connection with invariant subsets. On a finite dimensional space there does not exist a linear operator which is hypercyclic . But this is not the case with the bounded linear operators on infinite dimensional spaces. This was first observed by G.D. Birkhoff [4], who showed that the translation operator $f(z) \to f(z+1)$ is hypercyclic on the Fréchet space of all entire functions. Details on dynamical properties of operators can be found in [3].

Let $\mathcal{E}$ be a separable Hilbert space of complex valued functions on a nonempty set X and $\phi : X \to X$ be a map. The composition operator C_ϕ is defined by

$$(C_\phi f)(x) = f(\phi(x)) \ \text{ for all } f \in \mathcal{E}, \ x \in X.$$

Such operators are clearly linear. The basic idea in the study of composition operators is to describe the operator theoretic properties of C_ϕ with the help of function theoretic properties of ϕ and vice versa.

G. Ramesh: Department of Mathematics, Indian Institute of Technology - Hyderabad, Kandi, Sangareddy, Telangana, India 502 284, E-mail: rameshg@math.iith.ac.in
B. Sudip Ranjan: Department of Mathematics, Indian Institute of Technology - Hyderabad, Kandi, Sangareddy, Telangana, India 502 284, E-mail: ma16resch11003@iith.ac.in
D. Venku Naidu: Department of Mathematics, Indian Institute of Technology - Hyderabad, Kandi, Sangareddy, Telangana, India 502 284, E-mail: venku@math.iith.ac.in

Since nth-powers of the composition operator C_ϕ is related with the composition induced by the n^{th}-iterates of ϕ, the cyclic property of C_ϕ is connected with the dynamics of ϕ. In [8], Guo, Kunyu; Izuchi, Keiji gave a necessary and sufficient condition for a holomorphic mapping to be a cyclic vector of a composition operator on Fock type space.

In [9], Jiang, Liangying; Prajitura, Gabriel T.; Zhao, Ruhan gave a necessary and sufficient condition for the cyclicity of composition operator on the classical Fock space $\mathcal{F}^2(\mathbb{C})$. In [11], T. Mengestie proved that the cyclicity of weighted composition operator $C_{\psi,\varphi}$ on the classical Fock space $\mathcal{F}^2(\mathbb{C})$ depends on the inducing map $\varphi(z) = az + b$, where $|a| \leqslant 1$, $b \in \mathbb{C}$ and the weight function ψ.

In the remaining part of this section, we give brief details of the basic material that we need to prove our main results. In the second section, we study the cyclic behaviour of the composition operator C_ϕ and establish the connection between dynamical behaviour of C_ϕ and the Hilbert space operator A^*. In other words, we are able to show that the cyclic behaviour of a composition operator is strongly influenced by the dynamical properties of its inducing map, $\phi(z) = Az + b$ for $z \in \mathcal{E}$. In the third section, we show that there is no supercyclic composition operator on the Segal-Bargmann space under certain conditions on the inducing map. Later we discuss about the hypercyclic property.

1.1 The space $\mathcal{H}(\mathcal{E})$

Let $\mathcal{E}$ be an arbitrary infinite dimensional complex Hilbert space. For each integer $m \geqslant 1$, we write $\mathcal{E}^m$ for the symmetric tensor product of m copies of $\mathcal{E}$. Define $\mathcal{E}^0 = \mathbb{C}$ with its usual inner product, $\mathcal{E}^1 = \mathcal{E}$ and for $m \geqslant 2$, $\mathcal{E}^m$ is the closed subspace of the full tensor product $\mathcal{E}^{\otimes m}$ consisting of all elements that are invariant under the natural action of the symmetric group S_m. Precisely,

$$\mathcal{E}^m = \{x \in \mathcal{E}^{\otimes m} : \pi x = x \text{ for all } \pi \in S_m\}.$$

The action of S_m on $\mathcal{E}^{\otimes m}$ is defined on elementary tensors by

$$\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = x_{\pi(1)} \otimes \cdots \otimes x_{\pi(m)}.$$

For any $z \in \mathcal{E}$, we use $z^m = z \otimes \cdots \otimes z \in \mathcal{E}^m$ to denote the tensor product of m copies of z. Each $\mathcal{E}^m$ is a Hilbert space with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{E}^m}$ defined by

$$\langle z^m, w^m \rangle_{\mathcal{E}^m} = \langle z, w \rangle_{\mathcal{E}}^m,$$

where $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ denotes the inner product on $\mathcal{E}$.

Definition 1.1. A function $p_m : \mathcal{E} \to \mathbb{C}$ is called continuous m-homogeneous polynomial on $\mathcal{E}$ if there exists an element $\zeta \in \mathcal{E}^m$ such that $p_m(z) = \langle z^m, \zeta \rangle$ for $z \in \mathcal{E}$.

Definition 1.2. A function $f : \mathcal{E} \to \mathbb{C}$ is called continuous polynomial if f can be written as a finite sum of continuous homogeneous polynomials. That is, there is an integer $m \geqslant 0$ and there are elements $a_j \in \mathcal{E}^j$, $j = 0, 1, \cdots, m$ such that

$$f(z) = \sum_{j=0}^{m} p_j(z) = \sum_{j=0}^{m} \langle z^j, a_j \rangle.$$

We denote the space of all continuous m-homogeneous polynomials and the space of all continuous polynomials on $\mathcal{E}$ by $\mathcal{P}_m(\mathcal{E})$ and $\mathcal{P}(\mathcal{E})$, respectively. For more about m-homogeneous polynomials we refer to [12, p. 12].

For f, g in $\mathcal{P}(\mathcal{E})$, we can find an integer $m \geqslant 0$ and elements $a_j, b_j \in \mathcal{E}^j$ for $0 \leqslant j \leqslant m$ such that

$$f(z) = \sum_{j=0}^{m} \langle z^j, a_j \rangle \qquad \textit{and} \qquad g(z) = \sum_{j=0}^{m} \langle z^j, b_j \rangle.$$

Define,

$$\langle f, g\rangle = \sum_{j=0}^{m} j!\langle b_j, a_j\rangle. \tag{1.1}$$

Then $\langle, \cdot, \rangle$ defines an inner product on $\mathcal{P}(\mathcal{E})$. The completion of $\mathcal{P}(\mathcal{E})$ in the norm induced by the above inner product is called the Segal-Bargmann space and it is denoted by $\mathcal{H}(\mathcal{E})$. For more details see [10, Section 2.1].

Proposition 1.3. [10] Each element f in $\mathcal{H}(\mathcal{E})$ can be identified as an entire function on $\mathcal{E}$ having a power series expansion of the form

$$f(z) = \sum_{j=0}^{\infty}\langle z^j, a_j\rangle \qquad \text{for all } z \in \mathcal{E},$$

where $a_j \in \mathcal{E}^j,\ j = 0, 1, 2, \ldots$. Furthermore, $\|f\|^2 = \sum_{j=0}^{\infty} j!\, \|a_j\|^2$.

Conversely, if $\sum_{j=0}^{\infty} j!\, \|a_j\|^2 < \infty$, then the power series $\sum_{j=0}^{\infty}\langle z^j, a_j\rangle$ defines an element in $\mathcal{H}(\mathcal{E})$.

Applying Proposition 1.3 to the function

$$K(z, w) := K_w(z) = \exp\langle z, w\rangle \text{ for all } z, w \in \mathcal{E},$$

we can say that this function is the reproducing kernel function for $\mathcal{H}(\mathcal{E})$ and the normalized kernel function is defined by

$$k_w(z) = \exp(\langle z, w\rangle - \frac{\|w\|^2}{2}).$$

The linear span of the set $\{K_w : w \in \mathcal{E}\}$ is dense in $\mathcal{H}(\mathcal{E})$. As a result, $\mathcal{H}(\mathcal{E})$ is a reproducing kernel Hilbert space. For each $f \in \mathcal{H}(\mathcal{E})$, we have $\langle f, K(\cdot, x)\rangle = f(x)$ for all $x \in \mathcal{E}$. For a general theory of these spaces, see Chapter 2 of [1] and [13].

1.2 Composition operators

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces and K_1 and K_2 be the kernel functions for $\mathcal{H}(\mathcal{E}_1)$ and $\mathcal{H}(\mathcal{E}_2)$ respectively. We denote K as the kernel function for both the spaces $\mathcal{H}(\mathcal{E}_1)$ and $\mathcal{H}(\mathcal{E}_2)$ since the kernel functions on these spaces have the same form.

For any mapping $\phi : \mathcal{E}_1 \to \mathcal{E}_2$, the composition operator $C_\phi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1)$ is defined by

$$C_\phi(h) = h \circ \phi \text{ for all } h \in \mathcal{H}(\mathcal{E}_2),$$

for which $h \circ \phi$ also belongs to $\mathcal{H}(\mathcal{E}_1)$. Since C_ϕ is a closed operator, it follows from the closed graph theorem that C_ϕ is bounded if and only if $h \circ \phi$ belongs to $\mathcal{H}(\mathcal{E}_1)$ for all $h \in \mathcal{H}(\mathcal{E}_2)$.

If C_ϕ is bounded, then we have the following identities:

1. $C_\phi^* K_z = K_{\phi(z)}$ for all $z \in \mathcal{E}_1$,
2. Let $a \in \mathcal{E}_2$. If $f(w) = \langle w, a\rangle$ for all $w \in \mathcal{E}_2$, then

$$(C_\phi f)(z) = \langle \phi(z), a\rangle \text{ for any } z \in \mathcal{E}_1.$$

Theorem 1.4. *[10, Theorem 1.3] Let $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ be a mapping. Then the composition operator $C_\phi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1)$ is bounded if and only if $\phi(z) = Az + b$ for all $z \in \mathcal{E}_1$, where $A : \mathcal{E}_1 \to \mathcal{E}_2$ is a bounded linear operator with $\|A\| \leqslant 1$ and A^*b belongs to the range of $(I - A^*A)^{\frac{1}{2}}$. Furthermore, the norm of $\|C_\phi\|$ is given by*

$$\|C_\phi\| = \exp\left(\frac{1}{2}\|v\|^2 + \frac{1}{2}\|b\|^2\right),$$

where v is the unique vector in $\mathcal{E}_1$ of minimum norm satisfying $A^*b = (I - A^*A)^{\frac{1}{2}} v$.

Theorem 1.5. *[10, Theorem 3.7] Let $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ be a mapping. Then the composition operator $C_\phi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1)$ is bounded if and only if there is a bounded linear operator $A : \mathcal{E}_1 \to \mathcal{E}_2$ with $\|A\| \leqslant 1$ and a vector b in the range of $(I - AA^*)^{\frac{1}{2}}$ such that $\phi(z) = Az + b$ for all $z \in \mathcal{E}_1$. Furthermore, the norm of $\|C_\phi\|$ is given by*

$$\|C_\phi\| = \exp\left(\frac{\|u\|^2}{2}\right),$$

where u is the unique vector in $\mathcal{E}_2$ of minimum norm that satisfies the equation $b = (I - AA^*)^{\frac{1}{2}} u$.

For $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{C}^n$, the boundedness and compactness of C_ϕ are discussed by Carswell, MacCluer, and Schuster in [6].

1.3 Cyclicity

Unless mentioned otherwise, we consider infinite dimensional complex separable Hilbert spaces which will be denoted by $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ etc.

Definition 1.6. [3, p. 1] A bounded linear operator $T : \mathcal{E} \to \mathcal{E}$ is said to be cyclic if there exists a non zero vector $x \in \mathcal{E}$ such that $\overline{\text{span}}\{T^n x : n \geqslant 0\} = \mathcal{E}$. In this case x is said to be a cyclic vector of T.

We call the set $\{T^n x : n \geqslant 0\}$ as the orbit of T and is denoted by $\text{Orb}(T, x)$. It may happen that the $\text{Orb}(T, x)$ itself is dense or the projective orbit is dense, without the linear span; in this case we have the following definitions.

Definition 1.7. [3, p. 1] A bounded operator $T : \mathcal{E} \to \mathcal{E}$ is said to be hypercyclic if $\text{Orb}(T, x)$ is dense in $\mathcal{E}$. In this case x is said to be a hypercyclic vector.

The operator T is said to be supercyclic if the set $\{\lambda T^n x : n \geqslant 0, \lambda \in \mathbb{C}\}$ is dense in $\mathcal{E}$.

In the classical Fock space $\mathcal{F}^2(\mathbb{C})$, the following is known:

Proposition 1.8. [9, Proposition 5.1] Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a holomorphic map given by $\varphi(z) = az + b$ with $|a| \leqslant 1$. Let $C_\varphi : \mathcal{F}^2(\mathbb{C}) \to \mathcal{F}^2(\mathbb{C})$ be the associated composition operator on the Fock space $\mathcal{F}^2(\mathbb{C})$. Then we have

(i) If $b = 0$ and $|a| = 1$, then C_φ is cyclic if and only if $a^n \neq a$ for every $n > 1$.
(ii) If $|a| < 1$ and $a \neq 0$, then C_ϕ is cyclic.

Definition 1.9. Let $T : \mathcal{E} \to \mathcal{E}$ be a bounded linear operator. The set $\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I : \mathcal{E} \to \mathcal{E}$ is invertible and $(T - \lambda I)^{-1}$ is bounded$\}$ is called the resolvent set and the complement $\sigma(T) = \mathbb{C} \backslash \rho(T)$ is called the spectrum of T.

It is well known that $\sigma(T)$ is a non empty compact subset of $\mathbb{C}$.

2 Cyclic properties of C_ϕ

In this section, we discuss the existence of a fixed point of the inducing map ϕ satisfying a few conditions. Here we show that for a non zero vector $z \in \mathcal{E}$, if the kernel function K_z is a cyclic vector for the bounded composition operator C_ϕ, then z is a cyclic vector for the bounded operator A^*. Next we also show that if C_ϕ^* cyclic, then the operator A is cyclic.

We say that a map ϕ on $\mathcal{E}$ has the property $\mathcal{P}$, if it satisfies the following:

(i) $\phi(z) = Az + b$ for all $z \in \mathcal{E}$

(ii) $A : \mathcal{E} \to \mathcal{E}$ is a bounded linear operator with $\|A\| \leqslant 1$ and $b \in \mathcal{E}$
(iii) A^*b belongs to the range of $(I - A^*A)^{\frac{1}{2}}$.

Remark 2.1. Let $\phi : \mathcal{E} \to \mathcal{E}$ be a mapping. Then ϕ satisfies the property $\mathcal{P}$ if and only if the induced composition operator C_ϕ is bounded on the Segal-Bargmann space $\mathcal{H}(\mathcal{E})$.

For any $z \in \mathcal{E}$, we have $C_\phi f(z) = f(\phi(z))$ for all $f \in \mathcal{H}(\mathcal{E})$. For any positive integer m, we have $C_\phi^m f(z) = f(\phi^m(z))$. Here ϕ^m denotes the m times composition of ϕ. So we have the following identity:

$$\phi^m(z) = A^m z + A^{m-1}b + \cdots + Ab + b, \text{ for all } z \in \mathcal{E}. \tag{2.1}$$

Therefore,

$$C_\phi^m f(z) = f\big(A^m z + A^{m-1}b + \cdots + Ab + b\big). \tag{2.2}$$

So, for any kernel function K_z, we have

$$\begin{aligned} C_\phi^m K_z(w) &= K_z\big(A^m w + A^{m-1}b + \cdots + Ab + b\big) \\ &= K_z\big(A^m w\big) K_z\big(A^{m-1}b + \cdots + Ab + b\big) \\ &= K_{(A^*)^m z}(w) K_z\big(A^{m-1}b + \cdots + Ab + b\big). \end{aligned} \tag{2.3}$$

From the above relation it is clear that the dynamical properties of the composition operator C_ϕ depend on the behaviour of the iterates of A.

Let us begin with the following lemma which will be useful in later part of this paper.

Lemma 2.2. *Let A be a bounded linear operator on $\mathcal{E}$ such that $\|A\| \leqslant 1$. Then* $\ker(I - A^*) \subset \ker(I - A^*A)^{\frac{1}{2}}$.

Proof. Suppose $v \in \ker(I - A^*)$ with $\|v\| = 1$, then $A^*v = v$. As $\|A\| \leqslant 1$, we have

$$1 \geqslant \|Av\| = \|Av\| \, \|v\| \geqslant |\langle Av, v\rangle| = |\langle A^*v, v\rangle||\langle v, v\rangle| = 1. \tag{2.4}$$

That is,

$$|\langle Av, v\rangle| = \|Av\| \, \|v\| \, . \tag{2.5}$$

This implies that $Av = \lambda v$ for some $\lambda \in \mathbb{C}$, and hence

$$\lambda = \langle \lambda v, v\rangle = \langle Av, v\rangle = \langle v, A^*v\rangle = \langle v, v\rangle = 1. \tag{2.6}$$

This shows that $Av = v$. From this we can get that $A^*Av = A^*v = v$, which implies that $v \in \ker(I - A^*A)^{\frac{1}{2}}$. Hence we get the desired conclusion. □

The next result describes the fixed point property of the symbol ϕ which induces a bounded composition operator C_ϕ on $\mathcal{H}(\mathcal{E})$.

Proposition 2.3. Let ϕ be a mapping on $\mathcal{E}$ satisfying the property $\mathcal{P}$. Then ϕ has a fixed point in each of the following cases:

1. A is compact with $\|A\| \leqslant 1$.
2. $1 \notin \sigma(A)$.

Proof. Note that C_ϕ is bounded and $\phi(z) = Az + b$ for all $z \in \mathcal{E}$. If $A = 0$ then $\phi(z) = b$ fixes the point $b \in \mathcal{E}$. Now assume that $A \neq 0$. Also note that if $b = 0$, then $\phi(z) = Az$ and 0 is the fixed point of ϕ.

Proof of (1):
Since A is compact, by [14, p. 68], we have $\operatorname{ran}(I - A)$ and $\operatorname{ran}(I - A^*A)^{\frac{1}{2}}$ both are closed. We claim that ϕ has a fixed point. That is, we show there exists a point $z_0 \in \mathcal{E}$ such that $Az_0 + b = z_0$. This means $b \in \operatorname{ran}(I - A)$. Since $\operatorname{ran}(I - A)$ is closed, to prove our claim it is enough to prove $b \in \ker(I - A^*)^\perp$.

Since ϕ satisfies $\mathcal{P}$, we have $A^*b \in \operatorname{ran}(I - A^*A)^{\frac{1}{2}} = (\ker(I - A^*A)^{\frac{1}{2}})^\perp \subset \ker(I - A^*)^\perp$, by Lemma 2.2. Hence there exists $z_0 \in \mathcal{E}$ such that $\phi(z_0) = z_0$, and this proves our claim. That is, ϕ fixes a point in $\mathcal{E}$.

Proof of (2):
$1 \notin \sigma(A)$ implies that $(I - A)$ is invertible, and in this case the fixed point is $(I - A)^{-1}b$. □

Remark 2.4. By (2) of Proposition 2.3, we can say that if $\|A\| < 1$, then ϕ has a fixed point in $\mathcal{E}$.

Corollary 2.5. *Let $\phi : \mathbb{C}^n \to \mathbb{C}^n$ be a mapping such that C_ϕ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$. Then ϕ has a fixed point.*

Proof. Proof follows by (1) of Proposition 2.3. □

The assumption in (2) of Proposition 2.3 that $1 \notin \ \sigma(A)$ cannot be removed and this is illustrated with the following example.

Example 2.6. Let μ be a real number such that $0 < \mu \leqslant 1$. Define the weighted unilateral shift on $\ell^2(\mathbb{N})$ by

$$S(x_1, x_2, x_3, \dots) = (0, \mu x_1, x_2, x_3, \dots), \forall \{x_n\} \in \ell^2(\mathbb{N}). \tag{2.7}$$

The adjoint S^* of S is given by

$$S^*(x_1, x_2, x_3, \dots) = (\mu x_2, x_3, x_4, \dots), \forall \{x_n\} \in \ell^2(\mathbb{N}). \tag{2.8}$$

Now we see that $S^*S \neq SS^*$.

The spectrum $\sigma(S)$ is given by $\sigma(S) = \{\lambda : |\lambda| \leqslant 1\}$. Observe that $1 \in \sigma(S)$.

Note that $S^*S(x_1, x_2, x_3, \dots) = (\mu^2 x_1, x_2, x_3, \dots), \forall \{x_n\} \in \ell^2(\mathbb{N})$. It can be seen that

$$(I - S^*S)(x_1, x_2, x_3, \dots) = ((1 - \mu^2)x_1, 0, 0, \dots), \forall \{x_n\} \in \ell^2(\mathbb{N}).$$

Hence

$$(I - S^*S)^{\frac{1}{2}}(x_1, x_2, x_3, \dots) = (\sqrt{(1 - \mu^2)}x_1, 0, 0, \dots), \forall \{x_n\} \in \ell^2(\mathbb{N}). \tag{2.9}$$

Let $\hat{b} = (1, \frac{\sqrt{1-\mu^2}}{\mu}, 0, 0, \dots)$. Then we have

$$(I - S^*S)^{\frac{1}{2}} e_1 = S^*\hat{b}, \tag{2.10}$$

where $e_1 = (1, 0, 0, \dots)$. Now consider the map $\hat{\psi} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by $\hat{\psi}(x) = Sx + \hat{b}$, for all $x \in \ell^2(\mathbb{N})$. Since $\|S\| = 1$, we conclude that $\hat{\psi}$ satisfies the property $\mathcal{P}$. Therefore, the corresponding composition operator $C_{\hat{\psi}}$ is bounded on $\mathcal{H}(\ell^2(\mathbb{N}))$. Explicitly the map $\hat{\psi}$ can be written as

$$\hat{\psi}(x_1, x_2, x_3, \dots) = (1, \mu x_1 + \frac{\sqrt{1 - \mu^2}}{\mu}, x_2, x_3, \dots), \forall \{x_n\} \in \ell^2(\mathbb{N}). \tag{2.11}$$

It can be easily verified that $\hat{\psi}$ has no fixed point in $\ell^2(\mathbb{N})$.

Theorem 2.7. *Let ϕ be a mapping on $\mathcal{E}$ satisfying the property $\mathcal{P}$ and has a fixed point. If $C_\phi^* : \mathcal{H}(\mathcal{E}) \to \mathcal{H}(\mathcal{E})$ is cyclic, then $A : \mathcal{E} \to \mathcal{E}$ is cyclic.*

Proof. Let $z_0 \in \mathcal{E}$ be a fixed point of ϕ. Then $\phi(z) = A(z - z_0) + z_0$ and so for all $n \geqslant 0$,

$$\phi^n(z) = A^n(z - z_0) + z_0 = (I - A^n)z_0 + A^n z.$$

Also, we have

$$(I - A^n)z_0 = A^{n-1}b + \cdots + Ab + b. \tag{2.12}$$

Note that

$$\begin{aligned} C_\phi^* K_w(z) = K_{\phi(w)}(z) &= e^{\langle z, Aw+b \rangle} \\ &= e^{\langle z, b \rangle} e^{\langle A^* z, w \rangle} \\ &= K_b(z) K_w(A^* z). \end{aligned} \tag{2.13}$$

Therefore, $C_\phi^* = M_{K_b} C_\tau$, where $\tau(z) = A^* z$ and M_{K_b} is multiplication by the kernel function K_b. Observe that $C_\phi^n = C_{\phi^n}$, where $\phi^n = \underbrace{\phi \circ \phi \cdots \circ \phi}_{n\text{-times}}$. Hence

$$(C_\phi^*)^n = C_{\phi^n}^* = M_{K_{(I-A^n)z_0}} C_{\tau^n}, \tag{2.14}$$

where $\tau^n(z) = (A^*)^n z$.

Let $f \in \mathcal{H}(\mathcal{E})$ be a cyclic vector for C_ϕ^*. As $f \in \mathcal{H}(\mathcal{E})$, we have

$$f(z) = \sum_{j=0}^{\infty} \langle z^j, a_j \rangle, \qquad a_j \in \mathcal{E}^j$$

with $\|f\|^2 = \sum_{j=0}^{\infty} j! \, \|a_j\|^2 < \infty$.

Next, we consider

$$\begin{aligned}
{C_\phi^*}^n f(z) &= K_{(I-A^n)z_0}(z) f((A^*)^n z) \\
&= \sum_{i=0}^{\infty} \langle z^i, \frac{((I-A^n)z_0)^i}{i!} \rangle \sum_{j=0}^{\infty} \langle z^j, (A^n)^{\otimes j} a_j \rangle \\
&= \Big(1 + \langle z, (I-A^n)z_0 \rangle + \langle z^2, \frac{((I-A^n)z_0)^2}{2} \rangle + \cdots \Big) \\
&\qquad \times \Big(a_0 + \langle z, A^n a_1 \rangle + \langle z^2, (A^n)^{\otimes 2} a_2 \rangle + \cdots \Big) \\
&= a_0 + \langle z, A^n a_1 + \bar{a}_0((I-A^n)z_0) \rangle + \cdots \\
&= a_0 + \langle z, A^n(a_1 - \bar{a}_0 z_0) + \bar{a}_0 z_0 \rangle + \cdots .
\end{aligned}$$

We claim that $a_1 - \bar{a}_0 z_0 \neq 0$. If not, then $a_1 = \bar{a}_0 z_0$. Hence

$$f(z) = a_0 + \langle z, \bar{a}_0 z_0 \rangle + \langle z^2, a_2 \rangle + \cdots \tag{2.15}$$

and

$${C_\phi^*}^n f(z) = a_0 + \langle z, \bar{a}_0 z_0 \rangle + \cdots . \tag{2.16}$$

Now choose an element $\eta \in \mathcal{E} \backslash \{0\}$ such that $\langle z_0, \eta \rangle = 0$ and consider the function $h(z) = \langle z, \eta \rangle$ for all $z \in \mathcal{E}$. Then we see that $\langle {C_\phi^*}^n f, h \rangle = a_0 \langle z_0, \eta \rangle = 0$ for all $n \geqslant 0$. This contradicts that f is a cyclic vector for C_ϕ^*. This proves the claim.

Now we claim that $v = -\bar{a}_0 z_0 + a_1$ is a cyclic vector of A. Let $\zeta \in \mathcal{E}$ such that

$$\langle A^n v, \zeta \rangle = 0 \qquad \text{for all } n \geqslant 0.$$

Define $H(z) = \langle z - z_0, \zeta \rangle$ for all $z \in \mathcal{E}$. Clearly $H \in \mathcal{H}(\mathcal{E})$. For $n \geqslant 0$,

$$\begin{aligned}
(C_{\phi^n} H)(z) = H(\phi^n(z)) &= \langle \phi^n(z) - z_0, \zeta \rangle \\
&= \langle A^n(z - z_0), \zeta \rangle = -\langle z_0, (A^*)^n \zeta \rangle + \langle z, (A^*)^n \zeta \rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\langle H, {C_\phi^*}^n f \rangle &= \langle {C_\phi}^n H, f \rangle \\
&= -\bar{a}_0 \langle z_0, (A^*)^n \zeta \rangle + \langle a_1, (A^*)^n \zeta \rangle = \langle A^n v, \zeta \rangle = 0.
\end{aligned}$$

Since the linear span of $\{(C_\phi^*)^n f : n \geqslant 0\}$ is dense in $\mathcal{H}(\mathcal{E})$, we conclude that $H(z) = 0$ for all $z \in \mathcal{E}$, which implies $\zeta = 0$. Therefore, the set $\{A^n v : n \geqslant 0\}$ is dense in $\mathcal{E}$, which means that v is the cyclic vector for A as required.

□

Corollary 2.8. *Let ϕ be a mapping on $\mathcal{E}$ satisfying the property $\mathcal{P}$ with $b = 0$. If $C_\phi : \mathcal{H}(\mathcal{E}) \to \mathcal{H}(\mathcal{E})$ is cyclic, then $A^* : \mathcal{E} \to \mathcal{E}$ is cyclic.*

Proof. Let $\phi(z) = Az$, then $C_\phi = (C_\psi)^*$, where $\psi(z) = A^*z$ and 0 is the fixed point, and hence by Theorem 2.7, the conclusion follows. □

The converse of the above result is not true and this is illustrated with the following examples:

Example 2.9. Let $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$, and $b = 0$. Then it is clear that both A, and A^* are cyclic (see [7, p. 86]) but the corresponding bounded composition operator C_ϕ is not cyclic on the Fock space $\mathcal{F}^2(\mathbb{C}^2)$, where $\phi(z) = Az$, and $\|A\| \leqslant 1$ as the matrix A is not invertible (see [9]).

Let $B = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$, and $b = 0$. We can easily verify that both B, and B^* are cyclic but the corresponding bounded composition operator C_ϕ is not cyclic on the Fock space $\mathcal{F}^2(\mathbb{C}^2)$, when $\phi(z) = Bz$, and $\|B\| \leqslant 1$ as the matrix B is not invertible (see [9]).

The following result is well known in the literature of composition operators. In fact this result is true for any composition operator. Since it is important in our context, we state it here.

Proposition 2.10. [3] Suppose the mapping $\phi : \mathcal{E} \to \mathcal{E}$ has two fixed points, then the bounded composition operator C_ϕ cannot be cyclic operator on $\mathcal{H}(\mathcal{E})$.

Proof. To prove this, we use the fact that the adjoint of a cyclic operator can have only simple eigenvalues (see [5, Proposition 2.7]). Suppose that there are two fixed points, namely α, and β. Then we have $C_\phi^* K_\alpha = K_{\phi(\alpha)} = K_\alpha$, and $C_\phi^* K_\beta = K_{\phi(\beta)} = K_\beta$. This shows that 1 is the eigenvalue for C_ϕ^* with multiplicity at least two, and hence C_ϕ cannot be cyclic. □

Example 2.11. Consider the operator D on $\ell^2(\mathbb{N})$, defined by

$$D(x_1, x_2, x_3, \dots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots), \text{ for all } \{x_n\} \in \ell^2(\mathbb{N}),$$

and the map $\phi(z) = Dz + e_2$, where $e_2 = (0, 1, 0, \dots)$. Then we see that $\|D\| = 1$, and $(I - DD^*)^{\frac{1}{2}}\tilde{x} = e_2$, where $\tilde{x} = (0, \sqrt{\frac{4}{3}}, 0, \dots)$. Hence the corresponding composition operator C_ϕ is bounded operator on $\mathcal{H}(\ell^2(\mathbb{N}))$. The elements of the form $(\alpha, 2, 0, 0, \dots) \in \ell^2(\mathbb{N})$, where $\alpha \in \mathbb{C}$ are the fixed points of the map ϕ defined on $\ell^2(\mathbb{N})$. Now consider the elements $p = (0, 2, 0, 0, \dots)$, $q = (1, 2, 0, 0, \dots) \in \ell^2(\mathbb{N})$, and the kernel functions K_p, K_q in $\mathcal{H}(\ell^2(\mathbb{N}))$, then it is easy see that K_p, and K_q are the eigenvectors corresponding to the eigenvalue 1. This shows that 1 is the eigenvalue for C_ϕ^* with multiplicity at least two, and hence C_ϕ cannot be cyclic. This shows that D^* is cyclic but the associated composition operator C_ϕ is not cyclic.

Remark 2.12. The above example shows that if $\phi(z) = Az + b$ be a mapping on $\mathcal{E}$ that satisfies the property $\mathcal{P}$, and A^* is cyclic but the associated bounded composition operator C_ϕ need not be cyclic.

Observe that not every cyclic operator on a Hilbert space have a dense range, but there are certain composition operators which have a dense range. For example, composition operators on the Hardy space $H^2(\mathbb{D})$ have dense range [5, Theorem 1.4]. The problem of determining which composition operators have dense range is nontrivial. This forces us to investigate the proceeding theorem.

Theorem 2.13. *Let ϕ be a mapping on $\mathcal{E}$ satisfying the property $\mathcal{P}$ with $b = 0$. If C_ϕ is cyclic, then the following are true:*

1. $\ker(A) = \{0\}$. *Consequently, ϕ is injective.*
2. *range of C_ϕ is dense in $\mathcal{H}(\mathcal{E})$.*

Proof. **Proof of (1):** Since C_ϕ is cyclic, by Corollary 2.8, the operator A^* is cyclic. Therefore, the dimension of the orthogonal complement of range of A^* is at most one, that is, $\dim\ker(A) = \dim\operatorname{ran}(A^*)^\perp \leqslant 1$.

First we suppose that $\dim\ker(A) = 0$. Then $\ker(A) = \{0\}$. Hence ϕ is injective.

Now suppose that $\dim\ker(A) = 1$. Then there is a non zero vector x in $\mathcal{E}$ such that $\ker(A) = \operatorname{span}\{x\}$. Clearly the mapping ϕ is not injective. Hence there exists distinct pair of points $\eta \neq \zeta$ such that $\phi(\eta) = \phi(\zeta)$. This implies that $\eta - \zeta \in \ker(A)$, and hence $\eta - \zeta = \mathbb{C}x$. From this we observe that there are infinitely many such pair of distinct points. As a consequence the set $G = \{(z, w) : z \neq w, \phi(z) = \phi(w)\}$ is infinite.

For each pair $(\eta, \zeta) \in G$, consider the non zero function $f = K_\eta - K_\zeta$, then $C_\phi^* f = K_{\phi(\eta)} - K_{\phi(\zeta)} = 0$. This shows that the function f in the $\ker(C_\phi^*) = \operatorname{ran}(C_\phi)^\perp$. Hence $\operatorname{ran}(C_\phi)^\perp$ has infinite dimension as G is an infinite set. This is a contradiction as we know that the orthogonal complement of the range of a cyclic operator has dimension at most one. This completes the proof.

Proof of (2): Since C_ϕ is cyclic, by (1), we have $\ker(A) = \{0\}$. Hence $\operatorname{ran}(A^*)$ is dense.

Note that if $p_n \to p$, then

$$\lim_{k\to\infty} \langle K_{p_n}, f\rangle = \lim_{k\to\infty} f(p_n) = f(p) = \langle K_p, f\rangle \tag{2.17}$$

for every $f \in \mathcal{H}(\mathcal{E})$.

Therefore, we have K_{p_n} converges weakly to K_p. It is easy to verify that $\|K_{p_n}\| \to \|K_p\|$ as $n \to \infty$. In a complex Hilbert space, if $z_n \to z$ weakly and $\|z_n\| \to \|z\|$, then $\|z_n - z\| \to 0$. Hence $K_{p_n} \to K_p$ in norm.

Since range of A^* is dense in $\mathcal{E}$, for any $z \in \mathcal{E}$ there is a sequence $\{A^* x_n\}_{n=1}^\infty$ such that $A^* x_n$ converges to z in norm. That is, $\|A^* x_n - z\| \to 0$ and this will imply $\|A^* x_n\| \to \|z\|$ as $n \to \infty$. Also note that $\|K_{A^* x_n}\|^2 = \exp(\|A^* x_n\|^2) \to \exp(\|z\|^2) = \|K_z\|^2$ as $n \to \infty$. From Eq. (2.17), we have $K_{A^* x_n} \to K_z$ weakly. Thus $K_{A^* x_n} \to K_z$ in norm. Since

$$C_\phi K_{x_n} = K_{A^* x_n}, \tag{2.18}$$

we have for all $z \in \mathcal{E}$, $K_z \in \overline{\operatorname{ran}}(C_\phi)$. Hence $\overline{\operatorname{ran}}(C_\phi) = \mathcal{H}(\mathcal{E})$. □

In the forthcoming result, we show that if a kernel function K_w is a cyclic vector for the composition operator C_ϕ, where the inducing map $\phi : \mathcal{E} \to \mathcal{E}$ satisfies the property $\mathcal{P}$, then w is a cyclic vector for A^*, the adjoint of A.

Theorem 2.14. *Let ϕ be a mapping on $\mathcal{E}$ satisfying the property $\mathcal{P}$. Then we have the following:*

1. *If K_w is a cyclic vector for C_ϕ for some nonzero $w \in \mathcal{E}$, then w is a cyclic vector for A^*.*
2. *In the case of $b = 0$, if $f(z) = \sum_{j=0}^{\infty} \langle z^j, a_j\rangle \in \mathcal{H}(\mathcal{E})$ is a nonzero cyclic vector for C_ϕ, then $a_j \neq 0$ for all $j = 0, 1, 2, 3, \ldots$.*
3. *If f is a nonzero element in $\mathcal{H}(\mathcal{E})$ such that there exists $x \in \mathcal{E}$ such that $f(\phi^n(x)) = 0$ for every $n \geqslant 0$, then f cannot be a cyclic vector for C_ϕ.*

Proof. **Proof of (1):** Since K_w is a cyclic vector for C_ϕ, we have $\overline{\operatorname{span}}\{C_\phi^n K_w : n \geqslant 0\} = \mathcal{H}(\mathcal{E})$. Let p be a nonzero vector in $\mathcal{E}$ such that

$$\langle (A^*)^n w, p\rangle = 0 \qquad \text{for all } n \geqslant 0. \tag{2.19}$$

Denote $y_n = A^{n-1}b + \cdots + Ab + b$. Let $g = 1 - K_p \in \mathcal{H}(\mathcal{E})$. By using Eq. (2.3), we see that for all $n \geqslant 0$,

$$\begin{aligned}
\langle C_\phi^n K_w, g\rangle &= \langle K_{(A^*)^n w} K_w(y_n), 1 - K_p\rangle \\
&= K_w(y_n) - K_w(y_n) K_{(A^*)^n w}(p) \\
&= K_w(y_n) - K_w(y_n) e^{\langle p, (A^*)^n w\rangle} \\
&= 0 \qquad \text{by Eq. (2.19).}
\end{aligned} \tag{2.20}$$

Since the $\operatorname{span}\{C_\phi^n K_w : n \geqslant 0\}$ is dense in $\mathcal{H}(\mathcal{E})$, it follows that $g = 0$, which means $K_p = 1$, and hence $p = 0$. Therefore, the linear span of $\{(A^*)^n w : n \geqslant 0\}$ is dense in $\mathcal{E}$ and thus w is a cyclic vector for A^*.

Proof of (2): Since $b = 0$, we have $\phi(z) = Az$ with $\|A\| \leqslant 1$. Then

$$\begin{aligned} C_\phi^n f(z) = f(A^n z) &= \sum_{j=0}^{\infty} \langle (A^n z)^j, a_j \rangle \\ &= a_0 + \langle z, (A^*)^n a_1 \rangle + \langle z^2, ((A^*)^n)^{\otimes 2} a_2 \rangle + \cdots . \end{aligned} \tag{2.21}$$

For the purpose of obtaining a contradiction, assume that $a_k = 0$ for some k. Then $C_\phi^n f(z) = \sum_{j=0; j\neq k}^{\infty} \langle z^j, ((A^*)^n)^{\otimes j} a_j \rangle$. Let $g \in \mathcal{H}(\mathcal{E})$ be any continuous k-homogeneous polynomial, then we see that $\langle C_\phi^n f, g \rangle = 0$ for all $n \geqslant 0$. This is contradiction to the fact that f is a cyclic vector for C_ϕ.

Proof of (3): To prove this, it suffices to show that $K_x \in \overline{\operatorname{span}}\{C_\phi^n f : n \geqslant 0\}^\perp$. To see this, consider

$$\begin{aligned} \langle \sum_{j=1}^{k} \alpha_j C_\phi^j f, K_x \rangle = \sum_{j=1}^{k} \alpha_j \langle C_\phi^j f, K_x \rangle &= \sum_{j=1}^{k} \alpha_j \langle C_{\phi^j} f, K_x \rangle \\ &= \sum_{j=1}^{k} \alpha_j \langle f, C_{\phi^j}^* K_x \rangle \\ &= \sum_{j=1}^{k} \alpha_j f(\phi^j(x)) = 0. \end{aligned}$$

Thus we conclude that f cannot be a cyclic vector for C_ϕ. □

3 Supercyclicity and Hypercyclicity

We know that if a bounded linear operator on a separable Hilbert space is hypercyclic then it is supercyclic. In [9, Theorem 5.4], it is shown that there is no bounded supercyclic (hence no hypercyclic) composition operator on $\mathcal{F}^2(\mathbb{C}^n)$. In this section, we show this also holds true in the Segal-Bargmann space $\mathcal{H}(\mathcal{E})$ under some suitable conditions, where $\mathcal{E}$ is any infinite dimensional separable complex Hilbert space.

Theorem 3.1. *[2, Theorem 2.2] Let T be a bounded supercyclic operator on a Hilbert space H, and the set $\{T^n : n \in \mathbb{N}\}$ is uniformly bounded. Then for each $x \in H$,*

$$\lim_{n\to\infty} T^n x = 0.$$

Proposition 3.2. Let ϕ be a mapping on $\mathcal{E}$ satisfying the property $\mathcal{P}$. Then C_ϕ cannot be supercyclic operator in each of the following cases:

(i) $b = 0$.
(ii) $b \neq 0$, and $\|A\| < 1$.

Proof. If possible, let C_ϕ be supercyclic. By Theorem 1.5, we have $\phi(z) = Az + b$, where $A : \mathcal{E} \to \mathcal{E}$ is a bounded linear operator with $\|A\| \leqslant 1$ and $b \in \operatorname{ran}(I - AA^*)^{\frac{1}{2}}$.

Proof of (i): $b = 0$:
Then $\phi(z) = Az$ for all $z \in \mathcal{E}$. Therefore, $\phi^n(z) = A^n z$. Then by norm formula in Theorem 1.4, We have $\left\|C_\phi^n\right\|^2 = \|C_{\phi^n}\|^2 = 1$ for all $n \geqslant 1$. Hence we conclude that the set $\{C_\phi^n : n \in \mathbb{N}\}$ is uniformly bounded. But $C_\phi^n(1) = 1$. Hence by Theorem 3.1, C_ϕ cannot be supercyclic.

Proof of (ii): $b \neq 0$ and $\|A\| < 1$. For any integer $n \geqslant 1$, $C_\phi^n = C_{\phi^n}$, where $\phi^n = \phi \circ \cdots \circ \phi$ is the composition of n copies of ϕ. Since $\|A\| < 1$, by Proposition 2.3, there exists a point $z_0 \in \mathcal{E}$ such that $\phi(z_0) = z_0$. Then

$$(I - A^n) z_0 = A^{n-1} b + \cdots + Ab + b. \tag{3.1}$$

Therefore, $\phi^n(z) = A^n z + A^{n-1}b + \cdots + Ab + b = A^n z + (I - A^n)z_0$. By using the norm formula in Theorem 1.5, we have $\left\|C_\phi^n\right\|^2 = \|C_{\phi^n}\|^2 = e^{\|u_n\|^2}$, where u_n is the vector of smallest norm satisfying

$$(I - A^n)z_0 = (I - A^n(A^n)^*)^{\frac{1}{2}} u_n. \tag{3.2}$$

Since $\|A\| < 1$, the operator $(I - A^n(A^n)^*)^{\frac{1}{2}}$ is invertible. Therefore, u_n is uniquely determined by $u_n = (I - A^n(A^n)^*)^{-\frac{1}{2}}(I - A^n)z_0$. Thus,

$$\begin{aligned} \|u_m\| &\leqslant \left\|(I - A^n(A^n)^*)^{-\frac{1}{2}}\right\| \|(I - A^n)z_0\| \\ &\leqslant (1 - \|A\|^{2n})^{-\frac{1}{2}} \|(I - A^n)z_0\| \\ &\leqslant \frac{2\|z_0\|}{\sqrt{1 - \|A\|}}. \end{aligned}$$

Hence we have $\left\|C_\phi^n\right\| \leqslant e^{\frac{\|z_0\|}{\sqrt{1-\|A\|}}}$.

Hence we conclude that the set $\{C_\phi^n : n \in \mathbb{N}\}$ is uniformly bounded. But $C_\phi^n(1) = 1$. Hence by Theorem 3.1, C_ϕ cannot be supercyclic.

□

In the preceding result, we have seen that for a mapping $\phi(z) = Az + b$ with $\|A\| < 1$ such that the composition operator C_ϕ is bounded operator on $\mathcal{H}(\mathcal{E})$, then C_ϕ cannot be supercyclic operator. The only remaining case is $\|A\| = 1$ and $b \neq 0$. In this case we are not able to settle down the supercyclic behavior of C_ϕ.

Next we can ask the following question.

Question 3.3. Let ϕ be a mapping on $\mathcal{E}$ satisfying the property $\mathcal{P}$. Is it true that C_ϕ is not supercyclic, whenever $b \neq 0$ and $\|A\| = 1$?

The following examples shows that there is a chance that C_ϕ need not be supercyclic in the case of $b \neq 0$ and $\|A\| = 1$.

Example 3.4. Consider the operator in Example 2.11. Then we see that $\|D\| = 1$, $b \neq 0$ and C_ϕ is not cyclic. Therefore, C_ϕ is not supercyclic composition operator on $\mathcal{H}(\ell^2(\mathbb{N}))$.

Example 3.5. Let us consider the map $\varphi(z) = Az + b$ on $\mathbb{C}^n$, where A is the $n \times n$ matrix with $\|A\| = 1$. Then the corresponding bounded composition operator C_φ is not supercyclic on $\mathcal{F}^2(\mathbb{C}^n)$ (see, [9, Theorem 5.4]). In fact there is no supercyclic bounded composition operator on $\mathcal{H}(\mathcal{E})$ whenever $\mathcal{E} = \mathbb{C}^n$.

3.1 Hypercyclicity

In the remaining part of this section we discuss hypercyclicity of C_ϕ on $\mathcal{H}(\mathcal{E})$.

Theorem 3.6. *[3, Proposition 1.17] If T is a bounded hypercyclic operator on an infinite dimensional complex separable Hilbert space H, then the point spectrum $\sigma_p(T^*) = \varnothing$.*

Proposition 3.7. Let $\phi(z) = Az + b$ be a mapping on $\mathcal{E}$ that satisfies the property $\mathcal{P}$. Then C_ϕ is not hypercyclic if ϕ has a fixed point.

Proof. Suppose that ϕ has a fixed point. That is, there is some $\alpha \in \mathcal{E}$ such that $\phi(\alpha) = \alpha$. Then we have for every $f \in \mathcal{H}(\mathcal{E})$

$$\langle f, C_\phi^* K_\alpha \rangle = \langle f, K_{\phi(\alpha)} \rangle = \langle f, K_\alpha \rangle.$$

This shows that the kernel function K_α is an eigenvector of C_ϕ^* corresponding to the eigenvalue 1. Thus by Theorem 3.6, we conclude that C_ϕ is not hypercyclic. □

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