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Applications of TP_2 Functions in Theory of Stochastic Orders: A Review of some Useful Results

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Abstract. In the literature on Statistical Reliability Theory and Stochastic Orders, several results based on theory of TP_2/RR_2 functions have been extensively used in establishing various properties. In this paper, we provide a review of some useful results in this direction and highlight connections between them.

Keywords. Hazard Rate Order, Likelihood Ratio Order, Reversed Hazard Rate Order, RR_2, TP_2

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1 Introduction

Due to its vast applicability in various areas of Statistics, Mathematics and Economics, theory of total positivity has received considerable attention in the literature. Among

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its many applications in different research domains, theory of total positivity is useful in deriving optimal decision procedures in Statistical Decision Theory, deciding optimal allocation of resources in Queuing and Reliability Theory, and determining optimal policies in Operations Management. In the field of Statistical Reliability Theory, totally positive (reverse regular) functions of order 2 (TP_2 (RR_2)) arise naturally in studying reliability of coherent systems. Significant contributions in this area have been made by Samuel Karlin who built its foundations in probability and statistical theory. In the year 1968, he wrote a classical book on this subject and inspired several new developments. Karlin (1968) defined a TP_2 (RR_2) function in the following way:

Definition 1.1. Let S_1 and S_2 be two subsets of the real line $\mathbb{R}$. A non-negative function $k : S_1 \times S_2 \to [0, \infty)$ is said to be TP_2 (RR_2) if, for $x_1 \le x_2$, $y_1 \le y_2$ ($x_i \in S_1$; $y_i \in S_2$, $i = 1, 2$),

$$k(x_1, y_1)\, k(x_2, y_2) \ \ge (\le)\ k(x_1, y_2)\, k(x_2, y_1). \tag{1.1}$$

Karlin (1968) has also defined totally positive functions of order r (TP_r functions), $r \in \{3, 4, \ldots\}$. Due to limited applicability of totally positive functions of order greater than 2 in the applications relevant to our paper, we concentrate only on TP_2/RR_2 functions here. TP_2 functions were also studied by Lehmann (1966) who referred to the corresponding condition (given in (1.1)) as positive likelihood ratio dependence.

Among numerous applications, an application of theory of TP_2/RR_2 functions in Statistical Reliability Theory or Theory of Stochastic Orders is witnessed in proving various preservation results where one is interested in knowing whether a reliability property (e.g., decreasing failure rate, increasing failure rate, etc.) or a stochastic order (e.g., hazard rate order, reversed hazard rate order, etc.) satisfied by a collection of distributions is preserved under the operations of integral with respect to a sigma finite measure (such as those arising for mixtures or convolutions of distributions). Due to their importance in various applications, several general results have been obtained in the literature. For an account of contributions in this direction, one may refer to Lehmann (1966), Karlin (1968), Keilson and Sumita (1982), Dasgupta and Sarkar (1984), Lynch et al. (1987), Bartoszewicz (1998), Durham et al. (1990), Joag-dev et al. (1995), Khaledi and Shaked (2010), Marshall et al. (2010), Dewan and Khaledi (2014), Khaledi (2014), Laradji (2015) and Misra and Naqvi (2018).

To gain an insight into these studies, consider random variables (r.v.s) T_1 and T_2 having Lebesgue probability density functions (p.d.f.s) $g_1(\cdot)$ and $g_2(\cdot)$, respectively. Let $\psi_i : \mathbb{X} \times \mathbb{R} \to [0, \infty)$, $i = 1, 2$, be given non-negative functions, where $\mathbb{X} \subseteq \mathbb{R}$. In many applications (e.g., see Examples 1.1 - 1.3 discussed below), one is interested in deriving sufficient conditions that guarantee the monotonicity of a function of the type

$$\psi(x) = \frac{E(\psi_2(x, T_2))}{E(\psi_1(x, T_1))} = \frac{\int_{-\infty}^{\infty} \psi_2(x, \theta) g_2(\theta) d\theta}{\int_{-\infty}^{\infty} \psi_1(x, \theta) g_1(\theta) d\theta}, \tag{1.2}$$

for those values of $x \in \mathbb{X}$ for which atleast one of the integrals in the definition of $\psi(x)$ above is positive (with the convention that, for any $a > 0$, $a/0 = \infty$ and for any $b \in \mathbb{R}, b < \infty$). Note that the function $\psi(x)$, defined in (1.2), is monotonically increasing (decreasing) if, and only if, the function

$$\psi_i^*(x) = \int_{-\infty}^{\infty} \psi_i(x, \theta) g_i(\theta) d\theta, \ (i, x) \in \{1, 2\} \times \mathbb{X}, \tag{1.3}$$

is TP_2 (RR_2) in $(i, x) \in \{1, 2\} \times \mathbb{X}$, i.e., if, and only if, the inequality

$$\psi_1^*(x_2)\psi_2^*(x_1) \leq (\geq) \ \psi_1^*(x_1)\psi_2^*(x_2), \tag{1.4}$$

holds, whenever $x_1 \leq x_2$, $x_i \in \mathbb{X}$, $i = 1, 2$. Thus, studying preservation of TP_2/RR_2 property under the operation of an integral may be useful in many applications. More generally, one may be interested in monotonicity of the function of the type

$$\eta(x) = \frac{\int_a^b \eta_2(x, \theta) d\theta}{\int_a^b \eta_1(x, \theta) d\theta} = \frac{\eta_2^*(x)}{\eta_1^*(x)}, \ x \in S = \{t \in \mathbb{X} : \eta_1^*(t) + \eta_2^*(t) > 0\}, \text{ say}, \tag{1.5}$$

where, for $-\infty \leq a < b \leq \infty$, $\eta_i : \mathbb{X} \times (a, b) \to [0, \infty)$, $i = 1, 2$, are given non-negative functions, and

$$\eta_i^*(x) = \int_a^b \eta_i(x, \theta) d\theta, \ x \in \mathbb{X}, i = 1, 2. \tag{1.6}$$

Clearly, $\eta(x)$ is monotonically increasing (decreasing) in $x \in S$, if, and only if, $\eta_i^*(x)$ is TP_2 (RR_2) in $(i, x) \in \{1, 2\} \times \mathbb{X}$, or equivalently, the inequality

$$\eta_1^*(x_2)\eta_2^*(x_1) \leq (\geq) \ \eta_1^*(x_1)\eta_2^*(x_2), \tag{1.7}$$

holds for any $x_1 \leq x_2$, $x_i \in \mathbb{X}$, $i = 1, 2$.

Now, we provide three specific examples where such studies are useful. For definitions of some of the notions used in these examples, one may refer to Definitions 2.1 and 2.2 introduced in the sequel.

Example 1.1. (**Preservation of likelihood ratio, hazard rate and reversed hazard rate orderings by mixtures**) Let $\{X_1(\theta) : \theta \in \Theta\}$ and $\{X_2(\theta) : \theta \in \Theta\}$ be two families of r.v.s, where $\Theta \subseteq \mathbb{R}$ is an interval and, for $\theta \in \Theta$, $X_i(\theta)$ has the Lebesgue p.d.f. $h_{i,\theta}(\cdot)$, survival function (s.f.) $\overline{H}_{i,\theta}(\cdot)$ and distribution function (d.f.) $H_{i,\theta}(\cdot)$, $i = 1, 2$. Let T_1 and T_2 be two r.v.s having the Lebesgue p.d.f.s $g_1(\cdot)$ and $g_2(\cdot)$, respectively, and the same distributional support Θ. Let X_i be a r.v. (called a mixture r.v.) having the p.d.f.

$$m_i(x) = \int_{-\infty}^{\infty} h_{i,\theta}(x) I_\Theta(\theta) g_i(\theta) d\theta, \ -\infty < x < \infty,$$

the s.f.

$$\overline{M}_i(x) = \int_{-\infty}^{\infty} \overline{H}_{i,\theta}(x) I_\Theta(\theta) g_i(\theta) d\theta, \ -\infty < x < \infty,$$

and the d.f.

$$M_i(x) = \int_{-\infty}^{\infty} H_{i,\theta}(x) I_\Theta(\theta) g_i(\theta) d\theta, \ -\infty < x < \infty,$$

where, for any set $A \subseteq \mathbb{R}$, $I_A(\cdot)$ denotes its indicator function. It is known in the literature that if $X_1(\theta) \overset{d}{=} X_2(\theta)$ $(= X(\theta)$, say), $\forall\ \theta \in \Theta$, $X(\theta_1) \leq_{lr}$ (respectively, $\leq_{hr}$ and $\leq_{rh}$) $X(\theta_2)$, whenever $\theta_1 \leq \theta_2$, and $T_1 \leq_{lr}$ (respectively, $\leq_{hr}$ and $\leq_{rh}$) T_2, then $X_1 \leq_{lr}$ (respectively, $\leq_{hr}$ and $\leq_{rh}$) X_2 (see, for example, Theorems 1.C.17, 1.B.14 and 1.B.52 in Shaked and Shanthikumar (2007)); here $\overset{d}{=}$ means equality in distribution. For generalizing these results to situations where, for $\theta \in \Theta$, $X_1(\theta)$ and $X_2(\theta)$ may not be identically distributed, one requires sufficient conditions that ensure the increasing behaviour of the function

$$\psi(x) = \frac{\int_{-\infty}^{\infty} \psi_2(x, \theta) g_2(\theta) d\theta}{\int_{-\infty}^{\infty} \psi_1(x, \theta) g_1(\theta) d\theta}, \ -\infty < x < \infty,$$

where $\psi_i(x, \theta) = h_{i,\theta}(x) I_\Theta(\theta)$ (respectively, $\psi_i(x, \theta) = \overline{H}_{i,\theta}(x)\ I_\Theta(\theta)$ and $\psi_i(x, \theta) = H_{i,\theta}(x)$ $I_\Theta(\theta)$), $(x, \theta) \in (-\infty, \infty) \times \Theta$, $i = 1, 2$. Equivalently, one requires a set of sufficient conditions under which the function $m_i(x)$ (respectively, $\overline{M}_i(x)$ and $M_i(x)$) is TP_2 in $(i, x) \in \{1, 2\} \times (-\infty, \infty)$.

Example 1.2. (**Closure of DFR property under the operation of mixtures**) Let $g(\cdot)$ be the Lebesgue p.d.f. with an interval support Θ and let $\{\overline{F}_\theta : \theta \in \Theta\}$ be a family of absolutely continuous s.f.s. Assume that, for each $\theta \in \Theta$, $\overline{F}_\theta$s have the same interval support I. Consider the mixture s.f.

$$\overline{H}(x) = \int_\Theta \overline{F}_\theta(x) g(\theta) d\theta = \int_{-\infty}^{\infty} \overline{F}_\theta(x) I_\Theta(\theta) g(\theta) d\theta, \ x \in \mathbb{R}.$$

It is well known that if each $\overline{F}_\theta(\cdot)$, $\theta \in \Theta$, is log-convex on I (i.e., each $\overline{F}_\theta$, $\theta \in \Theta$, has decreasing failure rate (DFR)) then $\overline{H}(\cdot)$ is log-convex on I (i.e., $\overline{H}(\cdot)$ has DFR). In other words, any class of DFR distributions is closed under mixtures (Barlow and Proschan (1975)). Note that $\overline{H}(\cdot)$ is log-convex in $x \in I$ if, and only if, for every $\Delta > 0$, $\overline{H}(x+\Delta)/\overline{H}(x)$ is increasing in $x \in I$ (see, for example, Pečarić et al. (1992)). Since this ratio is of the form $\psi(x)$ (or $\eta(x)$), defined in (1.2) (or (1.5)), any result on monotonicity of function $\psi(x)$ (or $\eta(x)$) can be used here.

Example 1.3. (**Closure of IFR property under the operation of convolutions**) Let $\overline{F}_1$ and $\overline{F}_2$ be two absolutely continuous s.f.s with Lebesgue p.d.f.s f_1 and f_2, respectively, and let

$$\overline{H}(x) = \int_{-\infty}^{\infty} \overline{F}_2(x-\theta) f_1(\theta) d\theta, \ x \in \mathbb{R},$$

be the convolution of $\overline{F}_1$ and $\overline{F}_2$. It is well-known that $\overline{H}$ is log-concave on $\mathbb{R}$ (i.e., $\overline{H}$ has increasing failure rate (IFR)) if $\overline{F}_1$ and $\overline{F}_2$ are log-concave on $\mathbb{R}$ (i.e., $\overline{F}_1$ and $\overline{F}_2$ have IFR). This implies that the IFR class of distributions is closed under convolutions. Further, note that $\overline{H}(x)$ is log-concave if, and only if, for any $\Delta > 0$, $\overline{H}(x)/\overline{H}(x+\Delta)$ is increasing in $x \in I = \{t \in \mathbb{R} : \overline{H}(x) + \overline{H}(x+\Delta) > 0\}$. Therefore, any result ensuring monotonicity of functions of the type $\psi(\cdot)$ (or $\eta(\cdot)$) can be employed here.

In this paper, we provide a review of various results, available in the literature, that provide sufficient conditions under which inequality (1.4) (or (1.7)) holds (equivalently, $\psi_i^*(x)$ (or $\eta_i^*(x)$) is TP_2 (RR_2) in $(i, x) \in \{1, 2\} \times \mathbb{X}$) and highlight connections between them. By doing so, we aim to provide a guideline to researchers, working in the relevant areas, on applicability of these results in different situations.

The outline of this article is as follows. In the next section (Section 2), we introduce some notations, recall definitions and properties of various stochastic orders and reliability notions relevant to our paper and discuss some auxillary results. In Section

3, we discuss various general results that are frequently used in the literature and also highlight connections between them. In Section 4, we discuss some results that are useful in situations where underlying functions have a special structure. Finally, the study is concluded in Section 5.

2 Notations, Definitions and some Auxillary Results

Throughout the sequel, $\mathbb{R}$ will denote the real line $(-\infty, \infty)$ and, for any positive integer p, $\mathbb{R}^p$ will denote the p-dimensional Euclidean space. For any real number $a > 0$, we use the convention that $a/0 = \infty$ and that $b < \infty$, $\forall\, b \in \mathbb{R}$. When we say that a function $\zeta : D \to \mathbb{R}$ (where $D \subseteq \mathbb{R}$) is increasing (decreasing), it means that it is non-decreasing (non-increasing). For r.v.s X_1 and X_2, we write $X_1 \stackrel{d}{=} X_2$ to indicate that X_1 and X_2 are identically distributed.

Let X_1 and X_2 be two r.v.s with absolutely continuous d.f.s F_1 and F_2; s.f.s $\overline{F}_1 = 1 - F_1$ and $\overline{F}_2 = 1 - F_2$; and Lebesgue densities f_1 and f_2, respectively. Let $l_i = \inf\{x \in \mathbb{R} : F_i(x) > 0\}$, $u_i = \sup\{x \in \mathbb{R} : \overline{F}_i(x) > 0\}$ and $\{x \in \mathbb{R} : f_i(x) > 0\} = (l_i, u_i)$, $i = 1, 2$. Let $r_i(x) = f_i(x)/\overline{F}_i(x)$, $x < u_i$, and $\tilde{r}_i(x) = f_i(x)/F_i(x)$, $x > l_i$, respectively, denote the hazard rate function and the reversed hazard rate function of X_i, $i = 1, 2$. We first provide definitions of main stochastic orders. For a comprehensive discussion on various stochastic orders, one may refer to Whitt (1988), Bergmann (1991), Muller and Stoyan (2002), Lai and Xie (2006), Shaked and Shanthikumar (2007), Mosler and Scarsini (2012), Tong (2012), Szekli (2012), Li and Li (2013), Belzunce et al. (2016).

Definition 2.1. Under the above set-up, the r.v. X_1 is said to be smaller than the r.v. X_2 in the

(a) usual stochastic order (denoted by $X_1 \leq_{\mathrm{st}} X_2$) if $E(\phi(X_1)) \leq E(\phi(X_2))$, for every increasing function $\phi : \mathbb{R} \to \mathbb{R}$ for which the underlying expectations exist; or equivalently, if $l_1 \leq l_2$, $u_1 \leq u_2$ and $\overline{F}_1(x) \leq \overline{F}_2(x)$, $\forall\, x \in (l_2, u_1)$, whenever $l_2 < u_1$;

(b) hazard rate order (denoted by $X_1 \leq_{\mathrm{hr}} X_2$) if $\overline{F}_1(y)\overline{F}_2(x) \leq \overline{F}_1(x)\overline{F}_2(y)$, $\forall -\infty < x \leq y < \infty$; or equivalently, if $l_1 \leq l_2$, $u_1 \leq u_2$ and $\overline{F}_2(x)/\overline{F}_1(x)$ is increasing in $x \in (l_2, u_1)$, whenever $l_2 < u_1$;

(c) reversed hazard rate order (denoted by $X_1 \leq_{\mathrm{rh}} X_2$) if $F_1(y)F_2(x) \leq F_1(x)F_2(y)$, $\forall -\infty < x \leq y < \infty$; or equivalently, if $l_1 \leq l_2$, $u_1 \leq u_2$ and $F_2(x)/F_1(x)$ is increasing in $x \in (l_2, u_1)$, whenever $l_2 < u_1$;

(d) likelihood ratio order (denoted by $X_1 \leq_{\text{lr}} X_2$) if $f_1(y)f_2(x) \leq f_1(x)f_2(y)$, $\forall -\infty < x \leq y < \infty$; or equivalently, if $l_1 \leq l_2$, $u_1 \leq u_2$ and $f_2(x)/f_1(x)$ is increasing in $x \in (l_2, u_1)$, whenever $l_2 < u_1$.

It is noteworthy that $X_1 \leq_{hr} X_2$ (respectively, $X_1 \leq_{rh} X_2$ and $X_1 \leq_{lr} X_2$) if, and only if, $\overline{F}_i(\theta)$ (respectively, $F_i(\theta)$ and $f_i(\theta)$) is TP_2 in $(i, \theta) \in \{1, 2\} \times \mathbb{R}$. It is also well known in the literature that:

$$X_1 \leq_{\text{lr}} X_2 \Longrightarrow X_1 \leq_{\text{hr}} X_2 \text{ and } X_1 \leq_{\text{rh}} X_2 \Longrightarrow X_1 \leq_{\text{st}} X_2.$$

The following definition will be useful in defining some reliability notions.

Definition 2.2. Let $-\infty \leq c < d \leq \infty$. A non-negative function $\eta : (c, d) \to [0, \infty)$ is said to be log-concave (log-convex) on (c, d) if, for any $\alpha \in (0, 1)$ and $x, y \in (c, d)$,

$$\eta(\alpha x + (1 - \alpha)y) \ \geq (\leq) \ (\eta(x))^{\alpha} \ (\eta(y))^{1-\alpha},$$

or equivalently, if $\forall \, \Delta \in [0, d - c]$, $\eta(t)/\eta(t + \Delta)$ is increasing (decreasing) in $t \in (c, d - \Delta)$.

It is easy to verify that the log-concavity of a non-negative function η on an interval $I \subseteq \mathbb{R}$ guarantees its log-concavity on whole $\mathbb{R}$. The same is not true for log-convex functions. Consequently, f_1 (respectively, $\overline{F}_1$ and F_1) is log-concave on (l_1, u_1) if, and only if, f_1 (respectively, $\overline{F}_1$ and F_1) is log-concave on $\mathbb{R}$. However, log-convexity of f_1 (or, $\overline{F}_1$ and F_1) on (l_1, u_1) is not extendable to whole $\mathbb{R}$.

Definition 2.3. A real valued function η defined on an interval $I \subseteq \mathbb{R}$ is said to have atmost one sign change and from negative (positive) to positive (negative), as x increases on I, if any one of the following three conditions is satisfied:

(i) $\eta(x) \geq 0$, $\forall \, x \in I$;

(ii) $\eta(x) \leq 0$, $\forall \, x \in I$;

(iii) if there exists $x_0 \in I$ such that $\eta(x_0) > (<) \ 0$, then $\eta(x) > (<) \ 0$, $\forall \, x \in [x_0, \infty) \cap I$.

The following well known result provides a characterization of monotone functions in terms of sign changes.

Proposition 2.1. *Let τ be a real-valued function defined on an interval $I \subseteq \mathbb{R}$. Then τ is increasing (decreasing) on I if, and only if, $\forall \, \lambda \in \mathbb{R}$ the function $\phi_\lambda(x) = \tau(x) - \lambda$, $x \in I$, has atmost one sign change and from negative (positive) to positive (negative) as x increases on I.*

The result stated below is an immediate consequence of the above proposition.

Proposition 2.2. *Let* $-\infty \leq a < b \leq \infty$, $\mathbb{X} \subseteq \mathbb{R}$, *and let* $\eta_i : \mathbb{X} \times (a,b) \to [0,\infty)$, $i = 1,2$, *be given non-negative functions. Then, the function*

$$\eta(x) = \frac{\int_a^b \eta_2(x,\theta)d\theta}{\int_a^b \eta_1(x,\theta)d\theta} = \frac{\eta_2^*(x)}{\eta_1^*(x)}, \text{ say,}$$

is increasing (decreasing) in $x \in S = \{t \in \mathbb{R} : \eta_1^*(t) + \eta_2^*(t) > 0\}$ *if, and only if, for any* $\lambda \in \mathbb{R}$, *the function*

$$\phi_\lambda(x) = \int_a^b \Big[\eta_2(x,\theta) - \lambda\eta_1(x,\theta)\Big]d\theta, \; x \in \mathbb{X},$$

has atmost one sign change and from negative (positive) to positive (negative) as x increases on $\mathbb{X}$.

In conjunction to Propositions 2.1 and 2.2, another result that has been extensively used in the literature for proving monotonicity of functions, such as $\psi(\cdot)$ and $\eta(\cdot)$ defined in (1.2) or (1.5), respectively, is the following result given in Karlin (1968, Theorem 11.2, pp 324-325).

Theorem 2.1. *Let* $A, B \subseteq \mathbb{R}$ *and let* $\tau_1 : A \times B \to [0,\infty)$ *be a* TP_2 *(*RR_2*) function. Let* μ *be a* σ*-finite measure and let* $\tau_2 : A \times B \to \mathbb{R}$ *be such that the following conditions are satisfied:*

(i)

$$\tau(x) = \int_B \tau_1(x,\theta)\,|\tau_2(x,\theta)|\,d\mu(\theta) < \infty, \; \forall\, x \in A,$$

and $\tau(x)$ *defines a continuous function of* $x \in A$*;*

(ii) $\forall\, x \in A$, $\tau_2(x,\theta)$ *has atmost one sign change and from negative to positive as* θ *increases on* B*;*

(iii) $\forall\, \theta \in B$, $\tau_2(x,\theta)$ *increases (decreases) in* $x \in A$.

Then, $\tau(x)$ *has atmost one sign change and from negative to positive (positive to negative) as x increases on A.*

For the special case when $\tau_2(x,\theta)$ does not depend on x (i.e., $\tau_2(x,\theta) \equiv \tau_2(\theta)$, $\forall\,(x,\theta) \in A \times B$, say), the above result is stated as Theorem 3.5 (pp. 93) in Barlow and Proschan (1975) and is useful in proving preservation of various stochastic orders under general shock models.

3 Some General Results

In this section, we will discuss various results that provide sufficient conditions under which the function $\psi_i^*(x)$, defined in (1.3), is TP_2 (RR_2) in $(i, x) \in \{1,2\}\times\mathbb{X}$, or equivalently, the function $\psi(x)$, defined in (1.2), increases (decreases) for those values of $x \in \mathbb{X}$ for which atleast one of the integrals in (1.2) is positive.

First, we state the following result that follows from Theorem 5.1 given in Karlin (1968, pp. 123); also see Dewan and Khaledi (2014). Karlin's proof is based on the routine method of breaking involved integrals into two parts and then interchanging the variables of integration in one of the integrals. In order to gain insight into results based on sign changes of a function, presented in Propositions 2.1 and 2.2 and Theorem 2.1, we provide here a proof based on these results.

Theorem 3.1. *Let $-\infty \le a < b \le \infty$ and let $\eta_i : \mathbb{X}\times(a,b) \to [0,\infty), i = 1,2,$ be given non-negative functions, where $\mathbb{X} \subseteq \mathbb{R}$. Suppose that, for any fixed $x \in \mathbb{X}$, $\eta_i(x,\theta)$ is TP_2 in $(i,\theta) \in \{1,2\}\times(a,b)$ and, for every fixed $\theta \in (a,b)$, $\eta_i(x,\theta)$ is TP_2 (RR_2) in $(i,x) \in \{1,2\}\times\mathbb{X}$. In addition, suppose that either $\eta_1(x,\theta)$ or $\eta_2(x,\theta)$ is TP_2 (RR_2) in $(x,\theta) \in \mathbb{X}\times(a,b)$. Then the function $\eta_i^*(x)$, defined in (1.6), is TP_2 (RR_2) in $(i,x) \in \{1,2\}\times\mathbb{X}$.*

Proof. Fix $x \in \mathbb{R}$. In view of Proposition 2.2, it suffices to show that, for any $\lambda \in \mathbb{R}$, the function

$$\begin{aligned}\eta(x) &= \int_a^b [\eta_2(x,\theta) - \lambda\eta_1(x,\theta)]d\theta \\ &= \int_a^b \eta_1(x,\theta)\left[\frac{\eta_2(x,\theta}{\eta_1(x,\theta)} - \lambda\right]d\theta,\end{aligned}$$

has atmost one sign change and from negative to positive (positive to negative) as x increases on $S = \{x \in \mathbb{X} : \int_a^b \eta_1(x,\theta)d\theta + \int_a^b \eta_2(x,\theta)d\theta > 0\}$. This follows on using Theorem 2.1 along with the hypothesis of the theorem by taking $\tau_1 \equiv \eta_1$ and $\tau_2 \equiv \frac{\eta_2}{\eta_1}$. □

Now, we present results on monotonicity of function $\psi(x)$, defined in (1.2), for the special case when $\psi_1 \equiv \psi_2$. The following result is immediate from Theorem 3.1.

Theorem 3.2. *Let $\psi_1 : \mathbb{X}\times\mathbb{R} \to [0,\infty)$ be a given non-negative function. Suppose that*

(i) $g_i(\theta)$ is TP_2 in $(i,\theta) \in \{1,2\}\times\mathbb{R}$ (or equivalently, $T_1 \le_{\mathrm{lr}} T_2$);

(ii) $\psi_1(x,\theta)$ is TP_2 (RR_2) in $(x,\theta) \in \mathbb{X}\times\mathbb{R}$;

(iii) $\int_{-\infty}^{\infty} \psi_1(x,\theta) g_i(\theta) d\theta < \infty$, $\forall\, x \in \mathbb{X}$, $i = 1, 2$.

Then, $\psi_i^*(x) = \int_{-\infty}^{\infty} \psi_1(x,\theta) g_i(\theta) d\theta$ *is* TP_2 *(*RR_2*) in* $(i, x) \in \{1, 2\} \times \mathbb{X}$.

It is worth mentioning here that results stated in Theorems 3.1 and 3.2 are equivalent in the sense that they can be used interchangeably where they are applicable.

Recall that if $g_i(\theta)$ is TP_2 in $(i,\theta) \in \{1,2\} \times \mathbb{R}$ (i.e., $T_1 \leq_{\text{lr}} T_2$) then each of $\overline{G}_i(\theta)$ and $G_i(\theta)$ are TP_2 in $(i,\theta) \in \{1,2\} \times \mathbb{R}$ (i.e., $T_1 \leq_{\text{hr}} T_2$ and $T_1 \leq_{\text{rh}} T_2$). Thus, one would be interested in versions of Theorem 3.2 where condition (i) of Theorem 3.2 is replaced by a weaker condition that $G_i(\theta)$ is TP_2 in $(i,\theta) \in \{1,2\} \times \mathbb{R}$ or $\overline{G}_i(\theta)$ is TP_2 in $(i,\theta) \in \{1,2\} \times \mathbb{R}$. Of course that may be possible with trade-off of condition (ii) of Theorem 3.2 by stronger conditions. In this direction, the following two results follow from Joag-dev et al. (1995).

Theorem 3.3. *Let* $\psi_1 : \mathbb{X} \times \mathbb{R} \to [0, \infty)$ *be a given non-negative function. Suppose that*

(i) $\overline{G}_i(\theta)$ *is* TP_2 *in* $(i,\theta) \in \{1,2\} \times \mathbb{R}$ *(i.e.,* $T_1 \leq_{\text{hr}} T_2$*);*

(ii) $\psi_1(x,\theta)$ *is* TP_2 *(*RR_2*) in* $(x,\theta) \in \mathbb{X} \times \mathbb{R}$*;*

(iii) $\psi_1(x,\theta)$ *is increasing in* $\theta \in \mathbb{R}$*, for every* $x \in \mathbb{X}$*;*

(iv) $\int_{-\infty}^{\infty} \psi_1(x,\theta) g_i(\theta) d\theta < \infty$*, for every* $x \in \mathbb{X}$, $i = 1, 2$.

Then, $\psi_i^*(x) = \int_{-\infty}^{\infty} \psi_1(x,\theta) g_i(\theta) d\theta$ *is* TP_2 *(*RR_2*) in* $(i, x) \in \{1, 2\} \times \mathbb{X}$.

Theorem 3.4. *Let* $\psi_1 : \mathbb{X} \times \mathbb{R} \to [0, \infty)$ *be a given non-negative function. Suppose that*

(i) $G_i(\theta)$ *is* TP_2 *in* $(i,\theta) \in \{1,2\} \times \mathbb{R}$ *(i.e.,* $T_1 \leq_{\text{rh}} T_2$*);*

(ii) $\psi_1(x,\theta)$ *is* TP_2 *(*RR_2*) in* $(x,\theta) \in \mathbb{X} \times \mathbb{R}$*;*

(iii) $\psi_1(x,\theta)$ *is decreasing in* $\theta \in \mathbb{R}$*, for every* $x \in \mathbb{X}$*;*

(iv) $\int_{-\infty}^{\infty} \psi_1(x,\theta) g_i(\theta) d\theta < \infty$*, for every* $x \in \mathbb{X}$, $i = 1, 2$.

Then, $\psi_i^*(x) = \int_{-\infty}^{\infty} \psi_1(x,\theta) g_i(\theta) d\theta$ *is* TP_2 *(*RR_2*) in* $(i, x) \in \{1, 2\} \times \mathbb{X}$.

The following generalization of Theorem 3.2 to situations where, ψ_1 and ψ_2 may not be the same, is immediate from Theorem 3.1 (also see Dewan and Khaledi (2014)).

Theorem 3.5. *Let* $\psi_i : \mathbb{X} \times \mathbb{R} \to [0, \infty)$, $i = 1, 2$*, be given non-negative functions. Suppose that*

(i) $g_i(\theta)$ *is* TP_2 *in* $(i, \theta) \in \{1, 2\} \times \mathbb{R}$.

(ii) *either* $\psi_1(x, \theta)$ *or* $\psi_2(x, \theta)$ *is* TP_2 *in* $(x, \theta) \in \mathbb{X} \times \mathbb{R}$;

(iii) *for every fixed* $\theta \in \mathbb{R}$, $\psi_i(x, \theta)$ *is* TP_2 *in* $(i, x) \in \{1, 2\} \times \mathbb{X}$;

(iv) *for every fixed* $x \in \mathbb{X}$, $\psi_i(x, \theta)$ *is* TP_2 *in* $(i, \theta) \in \{1, 2\} \times \mathbb{R}$.

Then, $\psi_i^*(x) = \int_{-\infty}^{\infty} \psi_i(x, \theta) g_i(\theta) d\theta$ *is* TP_2 *in* $(i, x) \in \{1, 2\} \times \mathbb{X}$.

Realizing that it will be useful to have similar generalizations of Theorems 3.3 and 3.4 to situations where the functions ψ_1 and ψ_2 may be different, Misra and Naqvi (2018) provided these generalizations as reported in the sequel.

Theorem 3.6. *Let* $\psi_i : \mathbb{X} \times \mathbb{R} \to [0, \infty)$, $i = 1, 2$, *be given non-negative functions. Suppose that, for every fixed* $x \in \mathbb{X}$, $\psi_i(x, \theta)$ *is* TP_2 *in* $(i, \theta) \in \{1, 2\} \times \mathbb{R}$ *and, for every fixed* $\theta \in \mathbb{R}$, $\psi_i(x, \theta)$ *is* TP_2 *(*RR_2*) in* $(i, x) \in \{1, 2\} \times \mathbb{X}$. *In addition, suppose that either of the following three set of conditions are satisfied:*

(i) $g_i(\theta)$ *is* TP_2 *in* $(i, \theta) \in \{1, 2\} \times \mathbb{R}$ *and*

$\psi_1(x, \theta)$ *or* $\psi_2(x, \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in \mathbb{X} \times \mathbb{R}$;

(ii) $\overline{G}_i(\theta)$ *is* TP_2 *in* $(i, \theta) \in \{1, 2\} \times \mathbb{R}$ *and*

$\psi_1(x, \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in \mathbb{X} \times \mathbb{R}$ *and is increasing in* $\theta \in \mathbb{R}$

or

$\psi_2(x, \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in \mathbb{X} \times \mathbb{R}$ *and is increasing in* $\theta \in \mathbb{R}$;

(iii) $G_i(\theta)$ *is* TP_2 *in* $(i, \theta) \in \{1, 2\} \times \mathbb{R}$ *and*

$\psi_1(x, \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in \mathbb{X} \times \mathbb{R}$ *and is decreasing in* $\theta \in \mathbb{R}$

or

$\psi_2(x, \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in \mathbb{X} \times \mathbb{R}$ *and is decreasing in* $\theta \in \mathbb{R}$.

Then, the function $\psi_i^*(x) = \int_{-\infty}^{\infty} \psi_i(x, \theta) g_i(\theta) d\theta$ *is* TP_2 (RR_2) *in* $(i, x) \in \{1, 2\} \times \mathbb{X}$.

It is to be noted that when the stronger stochastic ordering (i.e., likelihood ratio order) in (i) is replaced by a weaker stochastic ordering (i.e., hazard rate or reversed hazard rate order) in (ii) or (iii) then we need an extra condition for the same result to hold true. It is worth mentioning here that the result stated in Theorem 3.6 (i) is equivalent to that stated in Theorem 3.1. It is also evident that the above theorem generalizes/extends the results of Karlin (1968), Joag-dev et al. (1995), Khaledi and Shaked (2010) and Dewan and Khaledi (2014).

We would like to add a note of caution for using Theorems 3.1 and/or 3.6. In many applications, $\psi_i(x,\theta)$ (or, $\eta_i(x,\theta)$), $i = 1,2$, will be a p.d.f./s.f./d.f. and the underlying assumptions of Theorems 3.1 and/or 3.6 may require $\psi_i(x,\theta)$ (or, $\eta_i(x,\theta)$) to be a log-convex function of $\theta \in \mathbb{R}$, whereas what will be known is the log-convexity of $\psi_i(x,\theta)$ (or, $\eta_i(x,\theta)$) on distributional support of T_i, $1 = 1,2$. Note that the log-convexity of a p.d.f./s.f./d.f. on the support of the associated distribution is not extendable to whole $\mathbb{R}$, and thus, in such situations, one needs to be careful in verifying validity of assumptions of Theorems 3.1 and/or 3.6. The same is not true for log-concavity of p.d.f./s.f./d.f. as their log-concavity on distributional support implies their log-concavity on whole $\mathbb{R}$.

4 Some Results under Special Structures

Although Theorems 3.1 and/or 3.6 are quite useful in many situations, they have some limitations, especially when ψ_i and/or g_i (or, η_i) have some special structure. To see this, let (T_1, T_2) and (U_1, U_2) be pair of independent random vectors with T_i (U_i) having absolutely continuous s.f. $\overline{F}_i$ ($\overline{H}_i$) and the Lebesgue p.d.f f_i (h_i), $i = 1,2$. Suppose that the common support of $\overline{F}_1, \overline{F}_2, \overline{H}_1$ and $\overline{H}_2$ is $[0,\infty)$. It is well-known that if $\overline{F}_i$ and $\overline{H}_i$, $i = 1,2$, are log-concave (i.e., they have IFR), and $T_i \leq_{hr} U_i$, $i = 1,2$, then $T_1 + T_2 \leq_{hr} U_1 + U_2$, i.e.,

$$\psi(x) = \frac{\int_{-\infty}^{\infty} \overline{H}_1(x-\theta) h_2(\theta) d\theta}{\int_{-\infty}^{\infty} \overline{F}_1(x-\theta) f_2(\theta) d\theta},$$

is increasing for those values of $x \in \mathbb{X} = [0,\infty)$ for which atleast one of the integrals in the definition of $\psi(x)$ above is positive. One can easily check that the aforementioned assertion cannot be proved using Theorems 3.1 or 3.6 as $\overline{H}_1(x-\theta)/\overline{F}_1(x-\theta)$ cannot be simultaneously increasing in $x \in \mathbb{X}$ (for fixed $\theta \in \mathbb{R}$) and in $\theta \in \mathbb{R}$ (for fixed $x \in \mathbb{X}$). For such situations, the following result that follows from Karlin (1968, pp. 124, Theorem 5.2) may be useful.

Theorem 4.1. *Let $\psi_i : \mathbb{R} \to [0, \infty)$, $i = 1, 2$, be given non-negative functions such that $\psi_i(x)$ is TP_2 in $(i, x) \in \{1, 2\} \times \mathbb{R}$ and suppose that $T_1 \leq_{lr} T_2$ (i.e., $g_i(\theta)$ is TP_2 in $(i, \theta) \in \{1, 2\} \times \mathbb{R}$). Additionally, suppose that ψ_1, ψ_2, g_1 and g_2 are log-concave functions on $\mathbb{R}$ (equivalently, $\psi_i(x - \theta)$ and $g_i(x - \theta)$, $i = 1, 2$, are TP_2 in $(x, \theta) \in \mathbb{R}^2$). Then,*

(a) $\psi_i^*(x) = \int_{-\infty}^{\infty} \psi_i(x - \theta) g_i(\theta) d\theta$ is TP_2 in $(i, x) \in \{1, 2\} \times \mathbb{R}$, or equivalently, the function

$$\psi(x) = \frac{\int_{-\infty}^{\infty} \psi_2(x - \theta) g_2(\theta) d\theta}{\int_{-\infty}^{\infty} \psi_1(x - \theta) g_1(\theta) d\theta},$$

is increasing for those values of $x \in \mathbb{R}$ for which atleast one of the integrals in the definition of $\psi(x)$ above is positive;

(b) for every $\Delta > 0$, $\psi_i^*(x)/\psi_i^*(x + \Delta)$ is increasing for $x \in S = \{t \in \mathbb{R} : \psi_i^*(t) + \psi_i^*(t + \Delta) > 0\}$ (equivalently, ψ_i^* is log-concave on $\mathbb{R}$), $i = 1, 2$.

On carefully analyzing the second proof of Theorem 5.2 of Karlin (1968, pp. 124), one realizes that Theorem 4.1 can, in fact, be proved under weaker assumptions detailed below (see, for example, proof of Theorem 1.1 in Hu and Zhu (2001)). For this purpose, we first introduce some shifted stochastic orders (that are stronger than their counterparts in Definition 2.1) which have been studied in the literature (see, for example, Shanthikumar and Yao (1986), Nakai (1995), Brown and Shanthikumar (1998), Lillo et al. (2000), Lillo et al. (2001), Di Crescenzo and Longobardi (2001), Hu and Zhu (2001), Belzunce et al. (2002), Nanda et al. (2006), and Aboukalam and Kayid (2007)).

Definition 4.1. The r.v. X_1 is said to be smaller than the r.v. X_2 in the

(a) up shifted hazard rate order (denoted by $X_1 \leq_{\mathrm{hr}\uparrow} X_2$) if $X_1 - \Delta \leq_{\mathrm{hr}} X_2$, $\forall\ \Delta \geq 0$, or equivalently, if $l_1 \leq l_2$, $u_1 \leq u_2$ and $\overline{F}_2(t)/\overline{F}_1(t + \Delta)$ is increasing in $t \in (l_2, u_1 - \Delta)$, $\forall\ \Delta \in (0, u_1 - l_2)$, whenever $l_2 < u_1$;

(b) up shifted reversed hazard rate order (denoted by $X_1 \leq_{\mathrm{rh}\uparrow} X_2$) if $X_1 - \Delta \leq_{\mathrm{rh}} X_2$, $\forall\ \Delta \geq 0$, or equivalently, if $l_1 \leq l_2$, $u_1 \leq u_2$ and $F_2(t)/F_1(t + \Delta)$ is increasing in $t \in (l_2, u_1 - \Delta)$, $\forall\ \Delta \in (0, u_1 - l_2)$, whenever $l_2 < u_1$;

(c) up shifted likelihood ratio order (denoted by $X_1 \leq_{\mathrm{lr}\uparrow} X_2$) if if $X_1 - \Delta \leq_{\mathrm{lr}} X_2$, $\forall\ \Delta \geq 0$, or equivalently, if $l_1 \leq l_2$, $u_1 \leq u_2$ and $f_2(t)/f_1(t + \Delta)$ is increasing in $t \in (l_2, u_1 - \Delta)$, $\forall\ \Delta \in (0, u_1 - l_2)$, whenever $l_2 < u_1$.

Theorem 4.2. *Let ψ_1 and ψ_2 be given non-negative functions on $\mathbb{R}$. For any $\Delta \geq 0$, define $\psi^*_{1,\Delta}(x) = \psi_1(x+\Delta)$ and $\psi^*_{2,\Delta}(x) = \psi_2(x)$. Suppose that, for any $\Delta \geq 0$, $\psi^*_{i,\Delta}(x)$ is TP_2 in $(i, x) \in \{1, 2\} \times \mathbb{R}$ (or equivalently, $\psi_2(x)/\psi_1(x+\Delta)$ is increasing in $x \in \{t \in \mathbb{R} : \psi_1(t+\Delta)+\psi_2(t) > 0\}$) and $T_1 \leq_{lr\uparrow} T_2$. Then, the function $\psi^*_i(x) = \int_{-\infty}^{\infty} \psi_i(x-\theta)g_i(\theta)d\theta$ is TP_2 in $(i, x) \in \{1, 2\} \times \mathbb{R}$, or equivalently, the function*

$$\psi(x) = \frac{\int_{-\infty}^{\infty} \psi_2(x-\theta)g_2(\theta)d\theta}{\int_{-\infty}^{\infty} \psi_1(x-\theta)g_1(\theta)d\theta},$$

is increasing for those values of $x \in \mathbb{R}$ for which atleast one of the integrals in the definition of $\psi(x)$ above is positive.

It is easy to verify that the assumptions of Theorem 4.1 are stronger than those of Theorem 4.2. To see another limitation of Theorems 3.1 and 3.6, let $\{\overline{F}_\theta : \theta \in \Theta\}$ be a collection of log-convex (DFR) s.f.s, where Θ is an interval on $\mathbb{R}$ and, for every $\theta \in \Theta$, the support of $\overline{F}_\theta$ is $[0, \infty)$. Denote a r.v. corresponding to $\overline{F}_\theta$ by $X(\theta)$, $\theta \in \Theta$. Let g be a Lebesgue pd.f. supported on $[0, \infty)$. Define the s.f.

$$\overline{F}(x) = \int_{\Theta} \overline{F}_\theta(x)g(\theta)d\theta = \int_{-\infty}^{\infty} \overline{F}_\theta(x)I_\Theta(\theta)g(\theta)d\theta, \ x \in \mathbb{R}.$$

Then, it is well known that $\overline{F}$ is log-convex (DFR) on $[0, \infty)$, or equivalently, for every $\Delta \geq 0$, the function

$$\psi(x) = \frac{\overline{F}(x+\Delta)}{\overline{F}(x)} = \frac{\int_{-\infty}^{\infty} \overline{F}_\theta(x+\Delta)I_\Theta(\theta)g(\theta)d\theta}{\int_{-\infty}^{\infty} \overline{F}_\theta(x)I_\Theta(\theta)g(\theta)d\theta},$$

is increasing on $[0, \infty)$. One can check that for proving this known fact, both Theorems 3.1 and 3.6, are applicable and would require the following assumptions for ascertaining the increasing behaviour of x on $[0, \infty)$:

(i) $\forall\ \theta \in \Theta$, $\overline{F}_\theta(x)$ is log-convex on $[0, \infty)$;

(ii) $\forall\ \theta_2 > \theta_1$, $X(\theta_1) \leq_{hr} X(\theta_2)$ or, $\forall\ \theta_2 > \theta_1$, $X(\theta_2) \leq_{hr} X(\theta_1)$.

Clearly, assumption (ii) above is the additional assumption required by Theorems 3.1 and 3.6 to establish the increasing behaviour of x and therefore, Theorems 3.1 and 3.6, do not yield desired results here. The following result due to Artin (1931) (also see Karlin (1968, Section 8.E, pp. 152) and Marshall et al. (2010, pp. 52)) is useful in such situations.

Theorem 4.3. *Let $\Theta \subseteq \mathbb{R}$ and $-\infty \le a < b \le \infty$ and let $\psi_\theta : (a,b) \to \mathbb{R}$, $\theta \in \Theta$, be a family of non-negative functions such that, for every $\theta \in \Theta$, $\psi_\theta(x)$ is log-convex in $x \in (a,b)$. For a positive measure μ on $\mathbb{R}$, suppose that the integral*

$$\psi(x) = \int_\Theta \psi_\theta(x) d\mu(\theta),$$

converges for all $x \in (a,b)$. Then, $\psi(x)$ is log-convex on (a,b) or, equivalently, for every $\Delta \in [0, b-a)$, the function

$$\frac{\psi(x+\Delta)}{\psi(x)} = \frac{\int_\Theta \psi_\theta(x+\Delta) d\mu(\theta)}{\int_\Theta \psi_\theta(x) d\mu(\theta)},$$

is increasing in $x \in (a, b-\Delta)$.

Finally, we present a result due to Prékopa (1971) (also see Pečarić et al. (1992, Theorem 13.26, pp. 353)) on preservation of log-concavity by marginals. For this, we require the definition of log-concavity on $\mathbb{R}^2$.

Definition 4.2. A non-negative function $\tau : \mathbb{R}^2 \to [0,\infty)$ is said to be log-concave on $\mathbb{R}^2$ if, for every $\alpha \in (0,1)$, $(x_1, x_2) \in \mathbb{R}^2$ and $(y_1, y_2) \in \mathbb{R}^2$,

$$\tau(\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2) \ge (\tau(x_1,x_2))^\alpha (\tau(y_1,y_2))^{1-\alpha}.$$

Theorem 4.4. *Let $\tau : \mathbb{R}^2 \to [0,\infty)$ be a log-concave function. Then the function*

$$\eta(x) = \int_{-\infty}^{\infty} \tau(x,\theta) d\theta,$$

is a log-concave function of $x \in \mathbb{R}$, or, equivalently, for every $\Delta \ge 0$, the function

$$\psi_\Delta(x) = \frac{\eta(x)}{\eta(x+\Delta)} = \frac{\int_{-\infty}^{\infty} \tau(x,\theta) d\theta}{\int_{-\infty}^{\infty} \tau(x+\Delta,\theta) d\theta},$$

is increasing in $x \in S = \{t \in \mathbb{R} : \eta(t) + \eta(t+\Delta) > 0\}$.

It is worth mentioning here that the above theorem can be used to provide sufficient conditions under which mixture of IFR distributions is IFR (also see Lynch (1999)).

Since the results stated in Theorems 3.1, 3.6, 4.2, 4.3 and 4.4, above are strongest among similar results available in the literature (e.g., those given in Karlin (1968), Joagdev et al. (1995), Khaledi and Shaked (2010), Dewan and Khaledi (2014) and Misra and Naqvi (2018)) on monotonicity of functions of the type $\psi(x)$ (or, $\eta(x)$), defined in 1.2 (1.5), we would prescribe use of these theorems in various applications.

5 Conclusion

In this paper, we reviewed various results based on theory of TP_2/RR_2 functions that are widely applicable in theory of stochastic orders and reliability. We also provide a guideline to users for usefulness of various results in different situations. While reviewing these results, we realized that for many applications it would be useful to obtain general results of the following type:

(i) derive sufficient conditions, such as those in Theorem 3.6 (i)-(iii), for monotonicity of functions of the type:

$$\psi(x) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_2(x,\theta_1,\theta_2)g_1(\theta_1)g_2(\theta_2)d\theta_1 d\theta_2}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_1(x,\theta_1,\theta_2)g_1(\theta_1)g_2(\theta_2)d\theta_1 d\theta_2},\ x \in \mathbb{X};$$

(ii) find sufficient conditions for monotonicity of functions of the type:

$$\psi(x) = \frac{\int_{-\infty}^{\infty}\psi_2(x-\theta)g_2(\theta)d\theta}{\int_{-\infty}^{\infty}\psi_1(x-\theta)g_1(\theta)d\theta},$$

under $T_1 \leq_{hr\uparrow} (\geq_{hr\uparrow})\ T_2$ and/ or $T_1 \leq_{rh\uparrow} (\geq_{rh\uparrow})\ T_2$ (see Theorem 4.2);

(iii) obtain sufficient conditions (different from those provided by Prékopa (1971), Theorem 4.4) for log-concavity of the function:

$$\psi(x) = \int_{-\infty}^{\infty}\psi_1(x,\theta)g_1(\theta)d\theta,\ x \in \mathbb{X},$$

(see, for example, Lynch (1999)).

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