



Some comparative results on eigenvalues, pseudospectra and conditionspectra

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Abstract

Conditionspectrum measures the computational stability of solving a linear system. In this paper, ten comparative results involving ε -conditionspectrum are presented. All these theorems generalize a well known eigenvalue theorem and simultaneously compare with an appropriate pseudospectra result.

Keywords Eigenvalues · Spectrum · Pseudo-spectrum · Condition spectrum

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The idea of *condition spectrum* was conceived while trying to realize pseudospectra as a special case of generalized spectra defined by Ransford in Ref. [7]. The axiomatic approach of his study compelled S. H. Kulkarni and the author in Ref. [3], to modify slightly the underlying basic set so as to satisfy the axioms of Ransford. The newly defined condition spectrum in itself has interesting properties which were published in Refs. [2–5]. In this article, its connections with the usual spectrum (eigenvalue) and pseudospectrum are discussed. Only those results on condition spectrum that have exemplary analogy with the other two spectra are chosen for presentation.

Each theorem is arranged in triplet, labeled with the same number followed by relevant suffix. The first part is a result about the usual spectrum. The second part is a generalization of the first result using pseudospectrum, while the third part, the main contribution of this article, generalizes the first part result in terms of condition

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spectrum. For example, Theorem 1, 1Λ and 1σ , denotes the first theorem for usual spectrum, pseudospectrum and condition spectrum respectively. This arrangement lets the reader compare and comprehend the results conveniently.

Our organizing principle is that each condition spectrum result precisely reduces to the corresponding eigenvalue theorem when $\varepsilon = 0$. Note that all the Λ and σ theorems reduce to corresponding usual spectrum (eigenvalue) theorem, as ε tends to zero. The format of each condition spectrum result is similar to the pseudospectrum result for easy comparison. This arrangement of results essentially follows the article [1].

1 Introduction

Let A be an $N \times N$ matrix with complex entries., ie $A \in M_N(\mathbb{C})$. Let λ and z denote complex numbers.

Definition 1.1 (*Condition spectrum*) Let $0 < \varepsilon < 1$. The ε -condition spectrum of a matrix A is defined as

$$\sigma_\varepsilon(A) := \left\{ z \in \mathbb{C} : z - A \text{ is not invertible or } \|(z - A)^{-1}\| \|z - A\| \geq \frac{1}{\varepsilon} \right\}.$$

Equivalently we can write it as $\sigma_\varepsilon(A) := \left\{ z \in \mathbb{C} : \|(z - A)^{-1}\| \|z - A\| \geq \frac{1}{\varepsilon} \right\}$ with the convention that $\|(z - A)^{-1}\| \|z - A\| = \infty$ when $z - A$ is not invertible. The name of the spectrum and the range of ε are justified from the definition, as it related to the condition number of $A - z$.

For the sake of completeness we give the definition of pseudospectrum which is used to understand the behavior of non-normal matrices. The properties and applications are available in Ref. [9].

Definition 1.2 (*Pseudospectrum*) Let $\varepsilon > 0$. The ε -pseudospectrum of a matrix A is defined as

$$\Lambda_\varepsilon(A) := \left\{ z \in \mathbb{C} : z - A \text{ is not invertible or } \|(z - A)^{-1}\| \geq \frac{1}{\varepsilon} \right\}.$$

Consider solving the system of equations $Ax - \lambda x = b$. The eigenvalues of A are the points in the complex plane at which $A - \lambda$ is not invertible and hence deal with the uniqueness of the solution. The pseudospectra of A convey the points at which $\|(A - \lambda)^{-1}\|$ becomes very large and hence deal with the computational aspects of the solution. The condition spectra capture the points in the complex plane at which the condition number $\|A - \lambda\| \|(A - \lambda)^{-1}\|$ becomes huge and hence elaborate the computational stability aspect of deriving the solution.

It is proved in Ref. [3] that the condition spectrum is a non empty, compact, perfect set (no isolated points) and always contains the usual spectrum. In Ref. [4], a sufficient condition for a function to be almost multiplicative is given using

conditionspectrum. Here we list two results which will be referred often subsequently. The interested reader can refer [3] for the proofs.

Lemma 1.3 For every $0 < \varepsilon < 1$, $\sigma(A) \subseteq \sigma_\varepsilon(A)$.

Lemma 1.4 For every $0 < \varepsilon < 1$, $\sigma_\varepsilon(A)$ is compact.

The above results are proved in a general setting, when A is a Banach algebra element. In contrast, the proof of the following equivalent conditions mainly relies on finite dimensionality.

Theorem 1.5 Let $A \in M_N(\mathbb{C})$. The following sets are equal (and hence equal to the condition spectrum).

- (1) $\sigma_\varepsilon(A) = \{z \in \mathbb{C} : \|(z - A)^{-1}\| \|z - A\| \geq \varepsilon^{-1}\}$
- (2) $B = \{z \in \mathbb{C} : \exists u \in \mathbb{C}^N \text{ with } \|u\| = 1 \text{ s.t. } \|(z - A)u\| \leq \varepsilon \|(z - A)\|\}$
- (3) $C = \{z \in \mathbb{C} : z \in \sigma(A + E) \text{ for some } E \text{ with } \|E\| \leq \varepsilon \|(z - A)\|\}$

Proof First observe that eigenvalues of A are there in all the three sets. For that, suppose z is an eigenvalue then

- $z \in \sigma_\varepsilon(A)$ by the convention $\|z - A\| \|(z - A)^{-1}\| = \infty$.
- $z \in B$ by taking u as any normalized eigenvector corresponding to the eigenvalue z .
- $z \in C$ by taking $E = 0$.

As eigenvalues belongs to all sets, it is enough to prove, the elements of one set (excluding eigenvalues) belongs to other sets. The following three implications together establish the proof of the theorem.

(1 \Rightarrow 2): Suppose $z \in \sigma_\varepsilon(A)$ and $z \notin \sigma(A)$ then $\|z - A\| \|(z - A)^{-1}\| \geq \varepsilon^{-1}$. Since unit sphere in finite dimensional space is compact, there exists an element $u \in \mathbb{C}^N$ such that $\|u\| = 1$ and

$$\|(z - A)^{-1}u\| = \|(z - A)^{-1}\|$$

Define

$$\tilde{u} = (z - A)^{-1}u,$$

so that $\|\tilde{u}\| = \|(z - A)^{-1}\|$ and $(z - A)\tilde{u} = u$. Let $\hat{u} = \frac{\tilde{u}}{\|\tilde{u}\|}$, then

$$\|(z - A)\hat{u}\| = \frac{\|(z - A)\tilde{u}\|}{\|\tilde{u}\|} = \frac{\|u\|}{\|\tilde{u}\|} = \frac{1}{\|(z - A)^{-1}\|} \leq \varepsilon \|z - A\|$$

and hence $z \in B$.

(2 \Rightarrow 3): Suppose $z \in B$ then there exists $u \in \mathbb{C}^N$ with $\|u\| = 1$ such that

$$\|(z - A)u\| \leq \varepsilon \|(z - A)\|.$$

Let $v \in \mathbb{C}^N$ be a unit vector satisfying $(A - z)u = \hat{\varepsilon}v$ with $\hat{\varepsilon} \leq \varepsilon\|z - A\|$. Let w be a vector with $\|w\| = 1$ such that $w^*u = 1$. With such a w we can write

$$zu = Au - \hat{\varepsilon}vw^*u = (A - \hat{\varepsilon}vw^*)u$$

which means that $z \in \sigma(A + E)$ for $E = -\hat{\varepsilon}vw^*$ satisfying $\|E\| \leq \hat{\varepsilon}$.

(3 \Rightarrow 1): Suppose $z \in \sigma(A + E)$ for some E with $\|E\| \leq \varepsilon\|z - A\|$. Then there exists a unit vector $v \in \mathbb{C}^N$ such that $(A + E)v = zv$. By rearranging and inverting

$$\begin{aligned} zv &= Av + Ev \\ (z - A)v &= Ev \\ v &= (z - A)^{-1}Ev \end{aligned}$$

and thus we have

$$1 = \|v\| = \|(z - A)^{-1}Ev\| \leq \|(z - A)^{-1}\| \|E\| \leq \|(z - A)^{-1}\| \varepsilon \|z - A\|,$$

implies that $\|(z - A)^{-1}\| \|z - A\| \geq \varepsilon^{-1}$ and hence $z \in \sigma_\varepsilon(A)$. □

2 Results

Ten results that are generalizing the theorems of usual spectrum are given. Only proofs of conditionspectrum results, that is theorems with the suffix σ , are given. Proofs of the usual spectrum are available in any standard book on linear algebra and the proofs of pseudospectra results are available in Ref. [1]. Note that each condition spectrum result tends to the corresponding eigenvalue theorem in the limiting case as ε tends to 0. Throughout this section, A be an $N \times N$ matrix with complex entries, i.e. $A \in M_N(\mathbb{C})$.

Theorem 1 A is singular $\iff 0 \in \sigma(A)$.

Theorem 1 Λ $\|A^{-1}\| \geq \varepsilon^{-1} \iff 0 \in \Lambda_\varepsilon(A)$ when A is not singular.

Theorem 1 σ $\|A\| \|A^{-1}\| \geq \varepsilon^{-1} \iff 0 \in \sigma_\varepsilon(A)$ when A is not singular.

Proof Follows from definition of condition spectrum. □

The bounds for the spectral radius of the condition spectrum.

Theorem 2 $\lambda \in \sigma(A) \Rightarrow |\lambda| \leq \|A\|$

Theorem 2 Λ $\lambda \in \Lambda_\varepsilon(A) \Rightarrow |\lambda| \leq \|A\| + \varepsilon$

Theorem 2 σ $\lambda \in \sigma_\varepsilon(A) \Rightarrow |\lambda| \leq \frac{1 + \varepsilon}{1 - \varepsilon} \|A\|$

Proof Let $\lambda \in \sigma_\varepsilon(A)$. If $|\lambda| \leq \|A\|$, then clearly $|\lambda| \leq \frac{1 + \varepsilon}{1 - \varepsilon} \|A\|$.

Suppose $|\lambda| > \|A\|$ then $\lambda - A$ is invertible and $\|(\lambda - A)^{-1}\| \leq \frac{1}{|\lambda| - \|A\|}$. Using this in the definition of condition spectrum we have

$$1 \leq \varepsilon \|(\lambda - a)^{-1}\| \|\lambda - a\| \leq \varepsilon \frac{|\lambda| + \|A\|}{|\lambda| - \|A\|}.$$

On simplification,

$$|\lambda| \leq \frac{1 + \varepsilon}{1 - \varepsilon} \|A\|.$$

□

Theorem 3 A has N distinct eigenvalues $\Rightarrow A$ is diagonalizable.

Theorem 3 $\Lambda_\varepsilon(A)$ has N distinct components $\Rightarrow A$ is diagonalizable.

Theorem 3 $\sigma_\varepsilon(A)$ has N distinct components $\Rightarrow A$ is diagonalizable.

Proof Refer [3] for a proof.

□

There are two components in condition number. One is norm of the matrix and the other is norm of its inverse. The following inequality estimates the norm of the inverse.

Lemma 2.1 Let \mathcal{A} be a Banach algebra. Let $a, b \in \mathcal{A}$. If a is invertible in \mathcal{A} and b is not invertible, then

$$\frac{1}{\|a^{-1}\|} \leq \|a - b\|. \tag{1}$$

The next result gives a relation between the norm of the resolvent with condition spectrum.

Theorem 4 $\|(z - A)^{-1}\| \geq \frac{1}{d(z, \sigma(A))}$

Theorem 4 $\Lambda \quad \|(z - A)^{-1}\| \geq \frac{1}{d(z, \Lambda_\varepsilon(A)) + \varepsilon}$

Theorem 4 $\sigma \quad \|(z - A)^{-1}\| \geq \frac{1}{d(z, \sigma_\varepsilon(A)) + \frac{2\varepsilon}{1-\varepsilon} \|A\|}$

Proof If $z \in \sigma(A)$ then the inequality is immediate from the definition of condition spectrum. So assume $z \notin \sigma(A)$ that is $z - A$ is invertible. Since $\sigma_\varepsilon(A)$ is compact, we can choose $\lambda \in \sigma_\varepsilon(A)$ such that $|\lambda - z| = d(z, \sigma_\varepsilon(A))$. As $\lambda \in \sigma_\varepsilon(A)$ there exist a matrix E such that $\|E\| \leq \varepsilon \|\lambda - A\|$ such that $\lambda \in \sigma(A + E)$. This implies $\lambda - A - E$ is not invertible and so $z - (A + E + z - \lambda)$ also. Now letting $a = z - A$ and $b = z - (A + E + z - \lambda)$ in Lemma 2.1, we get

$$\begin{aligned} \frac{1}{\|(z-A)^{-1}\|} &\leq \|z-A - [z-(A+E+z-\lambda)]\| = \|E+z-\lambda\| \\ &\leq |z-\lambda| + \|E\| \\ &\leq d(z, \sigma_\varepsilon(A)) + \varepsilon \|\lambda - A\| \leq d(z, \sigma_\varepsilon(A)) + \frac{2\varepsilon}{1-\varepsilon} \|A\| \end{aligned}$$

Hence it establishes the required inequality. \square

The spectrum is invariant under similarity transformation. Similarly the condition spectrum is also preserved under certain similarity transformation. This result will be helpful in the computational designing of conditionspectra. We use the standard notation $\kappa(S) = \|S\| \|S^{-1}\|$ to denote the condition number of a matrix S .

Theorem 5 $A = SBS^{-1} \Rightarrow \sigma(A) = \sigma(B)$

Theorem 5 Λ $A = SBS^{-1} \Rightarrow \Lambda_\varepsilon(A) \subseteq \Lambda_{\kappa(S)\varepsilon}(B)$

Theorem 5 σ $A = SBS^{-1} \Rightarrow \sigma_\varepsilon(A) \subseteq \sigma_{\kappa(S)^2\varepsilon}(B)$ whenever $\kappa(S)^2\varepsilon < 1$.

Proof Let $z \in \sigma_\varepsilon(A)$ then

$$\begin{aligned} \frac{1}{\varepsilon} \leq \|z-A\| \|(z-A)^{-1}\| &= \|zSS^{-1} - SBS^{-1}\| \|(zSS^{-1} - SBS^{-1})^{-1}\| \\ &\leq (\|S\| \|S^{-1}\|)^2 \|z-B\| \|(z-B)^{-1}\| \\ &\leq \kappa(S)^2 \|z-B\| \|(z-B)^{-1}\| \end{aligned}$$

\square

From this it is clear that similarity transformation through a matrix S with condition number 1, that is $\kappa(S) = 1$, preserves the condition spectrum, (since $\sigma_\varepsilon(A) \subseteq \sigma_\varepsilon(B)$ and $\sigma_\varepsilon(B) \subseteq \sigma_\varepsilon(A)$). In particular, similarity transformation through orthogonal and unitary matrices preserves the condition spectrum.

The next two results calculate the transient behavior of A from the knowledge of its condition spectrum. If one of the eigenvalue is bigger than 1 then the powers of A blows up to infinity. Condition spectrum also behaves in the same way.

Theorem 6

$$\max_{\lambda \in \sigma(A)} |\lambda| > 1 \Rightarrow \sup_{k \geq 0} \|A^k\| = \infty$$

Theorem 6 Λ

$$\max_{\lambda \in \Lambda_\varepsilon(A)} |\lambda| > 1 + M\varepsilon \Rightarrow \sup_{k > 0} \|A^k\| > M$$

Theorem 6 σ

$$\max_{\lambda \in \sigma_\varepsilon(A)} |\lambda| > \frac{1 + M^2\varepsilon}{1 - M\varepsilon} \Rightarrow \sup_{k \geq 0} \|A^k\| > M \quad \text{whenever } M < \frac{1}{\varepsilon}$$

Proof It is easy to establish the result for two simple and extreme cases. First one is the case when $M < 1$. Here the result is immediate as $\|A^0\| = 1$. The other case is when there is an eigenvalue λ of A with $|\lambda| > 1$. Since λ is an eigenvalue of A , λ^k is an eigenvalue of A^k for all $k \in \mathbb{N}$ and hence $|\lambda^k| \leq \|A^k\|$ for all k . This implies $\sup_{k \geq 0} \|A^k\|$ is infinity and hence the result.

Now, excluding the above mentioned simple cases, we prove the theorem by negation. That is, assume $M \geq 1$ and no eigenvalue of A has absolute value greater than 1. Suppose $\sup_{k \geq 0} \|A^k\| \leq M$, we will prove

$$\max_{\lambda \in \sigma_\varepsilon(A)} |\lambda| \leq \frac{1 + M^2\varepsilon}{1 - M\varepsilon}.$$

Let $\lambda \in \sigma_\varepsilon(A) \setminus \sigma(A)$. It is clear that if $|\lambda| \leq 1$ then $|\lambda| \leq \frac{1 + M^2\varepsilon}{1 - M\varepsilon}$ whenever $M < \frac{1}{\varepsilon}$. When $|\lambda| > 1$, as $\lambda - A$ is invertible, we get

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{|\lambda|} \sum_{k=0}^{\infty} \frac{\|A^k\|}{|\lambda|^k} \leq \frac{M}{|\lambda|} \sum_{k=0}^{\infty} \frac{1}{|\lambda|^k} = \frac{M}{|\lambda|} \left(\frac{1}{1 - \frac{1}{|\lambda|}} \right) = \frac{M}{|\lambda| - 1}.$$

Combining the above inequality with $\lambda \in \sigma_\varepsilon(A)$ gives $\frac{1}{\varepsilon\|\lambda - A\|} \leq \frac{M}{|\lambda| - 1}$. On simplification, $|\lambda| \leq \frac{1 + M^2\varepsilon}{1 - M\varepsilon}$ provided $M < \frac{1}{\varepsilon}$. As λ is arbitrary in $\sigma_\varepsilon(A) \setminus \sigma(A)$

$$\max_{\lambda \in \sigma_\varepsilon(A)} |\lambda| \leq \frac{1 + M^2\varepsilon}{1 - M\varepsilon},$$

and that proves the claim. □

Theorem 7 $\lambda \in \sigma(A) \Rightarrow \|A^k\| \geq |\lambda|^k$ for all k

Theorem 7 Λ $\lambda \in \Lambda_\varepsilon(A) \Rightarrow \|A^k\| \geq |\lambda|^k - \frac{k\varepsilon\|A\|^{k-1}}{1 - k\varepsilon/\|A\|}$ for all k such that $k\varepsilon < \|A\|$

Theorem 7 σ $\lambda \in \sigma_\varepsilon(A) \Rightarrow \|A^k\| \geq |\lambda|^k - \frac{ks\|A\|^{k-1}}{1 - ks/\|A\|}$ for all k such that $(2k + 1)\varepsilon < 1$ where $s = \frac{2\varepsilon}{1 - \varepsilon}\|A\|$

Proof Let $\lambda \in \sigma_\varepsilon(A)$. Pick $\|E\| \leq \varepsilon\|\lambda - A\|$ such that $\lambda \in \sigma(A + E)$. Then $|\lambda^k| \leq \|(A + E)^k\|$ which implies

$$\begin{aligned} \|A\|^k &\geq |\lambda|^k - \binom{k}{1} \|A\|^{k-1} \|E\| - \binom{k}{2} \|A\|^{k-2} \|E\|^2 - \dots \\ &\geq |\lambda|^k - k \|A\|^{k-1} \varepsilon \|\lambda - A\| \left(1 + \frac{k\varepsilon \|\lambda - A\|}{\|A\|} + \left(\frac{k\varepsilon \|\lambda - A\|}{\|A\|} \right)^2 + \dots \right). \end{aligned}$$

Provided $k\varepsilon \|\lambda - A\| \leq \|A\|$, the series in this last equation converges, giving

$$\|A\|^k \geq |\lambda|^k - \frac{k \|A\|^{k-1} \varepsilon \|\lambda - A\|}{1 - \frac{k\varepsilon \|\lambda - A\|}{\|A\|}} \geq |\lambda|^k - \frac{ks \|A\|^{k-1}}{1 - \frac{ks}{\|A\|}}$$

with $s = \frac{2\varepsilon}{1 - \varepsilon} \|A\|$. Also note that $(2k + 1)\varepsilon < 1$ implies $k\varepsilon \|\lambda - A\| \leq \|A\|$ by Theorem 2σ. □

The following result is analogous with Gerschgorin’s theorem which locate the spectrum in the complex plane. In our case it locates the condition spectrum. We denote the closed ball with center a and radius r by $D(a, r)$. Let d_j be the diagonal entry a_{jj} of A and $r_j = \sum_{i=1, i \neq j}^N |a_{ji}|$.

Theorem 8 For $A \in M_N(\mathbb{C})$, $\sigma(A) \subseteq \bigcup_{j=1}^N D(d_j, r_j)$

Theorem 8 Λ For $A \in M_N(\mathbb{C})$, $\Lambda_\varepsilon(A) \subseteq \bigcup_{j=1}^N D(d_j, r_j + \sqrt{N}\varepsilon)$

Theorem 8 σ For $A \in M_N(\mathbb{C})$, $\sigma_\varepsilon(A) \subseteq \bigcup_{j=1}^N D(d_j, r_j + \sqrt{N} \frac{2\varepsilon}{1-\varepsilon} \|A\|)$

Proof By equivalent definition, $\lambda \in \sigma_\varepsilon(A)$ if and only if $\lambda \in \sigma(A + E)$ for some E with $\|E\| \leq \varepsilon \|\lambda - A\|$. Applying the Gerschgorin’s theorem to $A + E$ we get

$$\lambda \in \bigcup_j D\left(d_j + e_{jj}, r_j + \sum_{k \neq j} |e_{jk}|\right)$$

where e_{ij} are the entries of E . The above balls are contained in balls centered at d_j with radius $r_j + \sum_{k=1}^N |e_{jk}|$. Now note that

$$\sum_{k=1}^N |e_{jk}| \leq \|E\|_\infty \leq \sqrt{N} \|E\|_2.$$

Alternatively one can deduce the above inequality by considering E_j as a matrix whose j th row is same as j th row of E and remaining rows are zero and observing

$$\sum_{k=1}^N |e_{jk}| \leq \|E_j\|_\infty \leq \sqrt{(N)} \|E_j\|_2 \leq \sqrt{N} \|E\|_2.$$

We know that $\|E\| \leq \varepsilon \|\lambda - A\| \leq \varepsilon \left(\frac{1+\varepsilon}{1-\varepsilon} \|A\| + \|A\| \right) \leq \frac{2\varepsilon}{1-\varepsilon} \|A\|$. Hence using this information and the previous inequality we get

$$\lambda \in \bigcup_j D\left(d_j, r_j + \sqrt{N} \frac{2\varepsilon}{1-\varepsilon} \|A\|\right)$$

Since λ is arbitrary, this proves the required result.

$$\sigma_\varepsilon(A) \subseteq \bigcup_j D\left(d_j, r_j + \sqrt{N} \frac{2\varepsilon}{1-\varepsilon} \|A\|\right)$$

□

Numerical range is equally explored like spectrum because of its computability. The next result connects numerical range and condition spectrum. We write $\text{conv}(S)$ for the convex hull in \mathbb{C} of a set $S \subseteq \mathbb{C}$. The notion $S \setminus \varepsilon$ -border means the set of points $z \in \mathbb{C}$ such that $D(z, \varepsilon) \subseteq S$

Theorem 9 $W(A) \supseteq \text{conv}(\sigma(A))$

Theorem 9 $\Lambda \quad W(A) \supseteq \text{conv}(\Lambda_\varepsilon(A)) \setminus \varepsilon$ -border

Theorem 9 $\sigma \quad W(A) \supseteq \text{conv}(\sigma_\varepsilon(A)) \setminus \varepsilon_1$ -border, here $\varepsilon_1 = \frac{2\varepsilon}{1-\varepsilon} \|A\|$.

Proof We first prove the following claim:

$$\text{conv}(\sigma_\varepsilon(A)) \subseteq W(A) + B\left(0, \frac{2\varepsilon}{1-\varepsilon} \|A\|\right).$$

If $\lambda \in \mathbb{C}$ and $d(\lambda, W(A)) > 0$, then $\lambda - A$ is invertible and (see [8, Theorem 6.2-A], also Ex. 27.7, p. 501, [6])

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{d(\lambda, W(A))}.$$

Let $\lambda \in \sigma_\varepsilon(A)$. If $\lambda \in \overline{W(A)}$, then the conclusion is obvious. Next, if $\lambda \in \sigma_\varepsilon(A) \setminus \overline{W(A)}$, then

$$\begin{aligned} d(\lambda, W(A)) &\leq \frac{1}{\|(\lambda - A)^{-1}\|} \leq \varepsilon \|(\lambda - A)\| \leq \varepsilon(|\lambda| + \|A\|) \\ &\leq \varepsilon\left(\frac{1+\varepsilon}{1-\varepsilon} \|A\| + \|A\|\right) = \frac{2\varepsilon}{1-\varepsilon} \|A\|. \end{aligned}$$

Thus

$$\sigma_\varepsilon(A) \subseteq W(A) + B\left(0, \frac{2\varepsilon}{1-\varepsilon} \|A\|\right).$$

Since the right hand side is a convex set we have established the claim. As $\text{conv}(\sigma_\varepsilon(A))$ and $W(A)$ are convex sets we get the required result. □

Spectral mapping theorem describes the behavior of the spectrum under certain transformation. The following result belongs to that kind. It gives a precise information about the condition spectrum under linear transformation.

Theorem 10 $\sigma(\alpha + \beta A) = \alpha + \beta\sigma(A)$ for all $\alpha, \beta \in \mathbb{C}$

Theorem 10 $\Lambda_{\varepsilon|\beta|}(\alpha + \beta A) = \alpha + \beta\Lambda_{\varepsilon}(A)$ for all $\alpha, \beta \in \mathbb{C}$

Theorem 10 $\sigma_{\varepsilon}(\alpha + \beta A) = \alpha + \beta\sigma_{\varepsilon}(A)$ for all $\alpha, \beta \in \mathbb{C}$

Proof For the case $\beta = 0$ follows from the fact $\sigma_{\varepsilon}(\alpha) = \{\alpha\}$. Consider $\beta \neq 0$,

$$\begin{aligned} \|z - (\alpha + \beta A)\| \|(z - (\alpha + \beta A))^{-1}\| &= \left\| \beta \frac{(z - \alpha)}{\beta} - \beta A \right\| \left\| \left(\beta \frac{(z - \alpha)}{\beta} - \beta A \right)^{-1} \right\| \\ &= \left\| \frac{(z - \alpha)}{\beta} - A \right\| \left\| \left(\frac{(z - \alpha)}{\beta} - A \right)^{-1} \right\| \end{aligned}$$

□

The above results analyse the similarity between condition spectra and pseudospectra. The results in Refs. [3, 4] presents basic properties of condition spectrum and its connection with other areas. All these results together demonstrate the potential of condition spectrum and the need for further investigation. The computational aspect of this spectrum is yet to be investigated. There are other results in Ref. [1] for which, at present, the generalizations in conditionspectrum are unknown.

Compliance with ethical standards

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

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