ON GENERATION OF THE COEFFICIENT FIELD OF A PRIMITIVE HILBERT MODULAR FORM BY A SINGLE FOURIER COEFFICIENT

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Abstract. For a primitive Hilbert modular form f over F of weight k, under certain assumptions on image of $\bar{\rho}_{f,\lambda}$, we calculate the Dirichlet density of primes $\mathfrak{p}$ for which the $\mathfrak{p}$-th Fourier coefficient $C(\mathfrak{p}, f)$ generates the coefficient field E_f. If $k = 2$, then we show that the assumption on the image of $\bar{\rho}_{f,\lambda}$ is satisfied when the degrees of E_f, F are equal and odd prime. We also compute the density of primes $\mathfrak{p}$ for which $C^*(\mathfrak{p}, f)$ generates F_f. Then, we provide some examples of f to support our results. Finally, we calculate the density of primes $\mathfrak{p}$ for which $C(\mathfrak{p}, f) \in K$ for any field K with $F_f \subseteq K \subseteq E_f$. This density is completely determined by the inner twists of f associated with K. This work can be thought of as a generalization of [KSW08] to primitive Hilbert modular forms.

1. Introduction

The study of the Fourier coefficients of modular forms is an active area of research in number theory. It is well-known that for any primitive form f over $\mathbb{Q}$, the Fourier coefficients of f generate a number field E_f. In [KSW08], the authors proved that the set of primes p for which p-th Fourier coefficient of f generates E_f has density one if f does not have any non-trivial inner twists. To the best of the author's knowledge, the analogous question is still open for Hilbert modular forms, which is the objective of our study in this article.

For a primitive form f over F, let E_f denote the number field generated by the Fourier coefficients $C(\mathfrak{p}, f)(\mathfrak{p} \in P)$ of f, where P denote the set of all prime ideals of F (cf. [Shi78]). We first state a result that, for a primitive form f over F of weight 2, the set of $\mathfrak{p} \in P$ for which $\mathbb{Q}(C(\mathfrak{p}, f)) = E_f$ has Dirichlet density 1, if $[F : \mathbb{Q}] = [E_f : \mathbb{Q}]$ is an odd prime (cf. Theorem 3.1). We then state and prove a general result for f of weight k (cf. Theorem 3.6) under some assumptions on the image of $\bar{\rho}_{f,\lambda}$(cf. (3.1))). We then show that these assumptions on the image of $\bar{\rho}_{f,\lambda}$ are satisfied for primitive forms f over F of weight 2, if $[F : \mathbb{Q}] = [E_f : \mathbb{Q}]$ is an odd prime. The proof of Theorem 3.1 mainly depends on an important proposition of Dimitrov (cf. [Dim05, Proposition 3.9]). We continue this study for the field $F_f \subseteq E_f$ and show that the set of $\mathfrak{p} \in P$ for which $\mathbb{Q}(C^*(\mathfrak{p}, f)) = F_f$ has density 1 (cf. §2.0.1 for the definitions of $C^*(\mathfrak{p}, f)$ and F_f).

This article builds on the ideas of Koo et al., in [KSW08], for primitive forms over $\mathbb{Q}$. One of the vital ingredients in the proof of [KSW08, Theorem 1.1] is a Theorem of Ribet (cf. [Rib85, Theorem 3.1]), where he explicitly described the image of l-adic

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residual Galois representation $\bar{\rho}_{f,l}$ attached to classical modular forms. This result plays a crucial role in obtaining certain sharp bounds for the images of $\bar{\rho}_{f,l}$, which are helpful in their proof. Unfortunately, in our context, an analog of Ribet's result does not seem to exist in the literature. In order to get similar sharp bounds for the images of $\bar{\rho}_{f,\lambda}$, we have to work with some assumptions (cf. (3.1) in the text). This explains the reason for our assumptions in Theorem 3.1 and Theorem 3.6.

Our results can be thought of as a generalization of the results in [KSW08] to primitive Hilbert modular forms f over F. Using LMFDB, we produce examples of primitive Hilbert modular forms f of parallel weight 2 in support of Theorem 3.1 (cf. Example 4, Example 5 and Example 6 in the text). Finally, we calculate the density of $\mathfrak{p} \in P$ for which $C(\mathfrak{p}, f) \in K$, where $K \subseteq E_f$ is a subfield. This density depends on whether $K \supseteq F_f$ or not. If $F_f \not\subseteq K$, then it is zero (cf. Lemma 4.1), otherwise it is non-zero and completely determined by the inner twists of f associated with K (cf. Proposition 4.3).

1.1. **Structure of the article:** The article is organized as follows. In §2, we collate all the preliminaries which are required to prove our main theorems (cf. Theorem 3.1, Theorem 3.6). We also introduce the notion of inner twists and study their properties quite elaborately. In §3, we state and prove Theorem 3.1 and its generalization i.e., Theorem 3.6 for primitive f over F of parallel weight 2 and weight k, respectively, under certain assumptions. We also prove a variant of these results for F_f and study their consequences. In §4, we calculate the Dirichlet density of $\mathfrak{p} \in P$ for which $C(\mathfrak{p}, f) \in K$ for any field K with $K \subseteq E_f$. This density is completely determined by the inner twists of f associated with K if $F_f \subseteq K$.

2. Preliminaries

Let F be a totally real number field. Let $\mathcal{O}_F$, $\mathfrak{n}$, and $\mathfrak{D}$ represent the ring of integers, an ideal, and the absolute different of F, respectively.

2.0.1. *Notations.* Throughout this article, we fix to use the following notations.

- Let $\mathbb{P}$, P denote the set of all primes in $\mathbb{Z}$, $\mathcal{O}_F$, respectively.
- Let $k = (k_1, k_2, ..., k_n)$ be an n-tuple of integers such that $k_i \geq 2$ and $k_1 \equiv k_2 \equiv \cdots \equiv k_n \pmod 2$. Let $k_0 := \max\{k_1, k_2, ..., k_n\}$, $n_0 = k_0 - 2$.
- For any number field K, denote $G_K := \mathrm{Gal}(\bar{K}/K)$. Let L be a subfield of K. For a prime ideal $\mathfrak{q}$ in K lying above $\mathfrak{p} = \mathfrak{q} \cap L$ in L, let $\mathrm{e}(\mathfrak{q}/\mathfrak{p})$ and $\mathrm{f}(\mathfrak{q}/\mathfrak{p})$ denote the ramification degree and inertia degree of $\mathfrak{q}$ over $\mathfrak{p}$, respectively.

For any Hecke character Ψ of F with conductor dividing $\mathfrak{n}$ and infinity-type $2-k_0$, let $S_k(\mathfrak{n}, \Psi)$ denote the space of all Hilbert modular newforms over F of weight k, level $\mathfrak{n}$ and character Ψ. A primitive form is a normalized Hecke eigenform in the space of newforms. The ideal character corresponding to Ψ of F is denoted by Ψ^*.

For a primitive form $f \in S_k(\mathfrak{n}, \Psi)$, let $C(\mathfrak{b}, f)$ denote the Fourier coefficient of f corresponding to an integral ideal $\mathfrak{b}$ of F and $C^*(\mathfrak{b}, f) := \frac{C(\mathfrak{b},f)^2}{\Psi^*(\mathfrak{b})}$ for all ideal $\mathfrak{b}$ with $(\mathfrak{b}, \mathfrak{n}) = 1$. Write $E_f = \mathbb{Q}(C(\mathfrak{b}, f))$, $F_f = \mathbb{Q}(C^*(\mathfrak{b}, f))$, where $\mathfrak{b}$ runs over all the integral ideals of $\mathcal{O}_F$ such that $(\mathfrak{b}, \mathfrak{n}) = 1$. Let $\mathcal{P}_f$ denote the set of all prime ideals in E_f. For any two subfields F_1, F_2 such that $\mathbb{Q} \subseteq F_2 \subseteq F_1 \subseteq E_f$, we let

$$\mathrm{f}_{\lambda, F_1, F_2} := \mathrm{f}(\lambda \cap F_1 / \lambda \cap F_2)$$

for $\lambda \in \mathcal{P}_f$. The following proposition describes some properties of E_f.

Proposition 2.1 ([Shi78]). *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form of weight k, level $\mathfrak{n}$ and character Ψ with coefficient field E_f. Then*

1. *E_f is a finite Galois extension of $\mathbb{Q}$,*
2. *$\Psi^*(\mathfrak{m}) \in E_f$, for all ideals $\mathfrak{m} \subseteq \mathcal{O}_F$,*
3. *E_f is either a totally real or a CM field,*
4. *$E_f = \mathbb{Q}(\{C(\mathfrak{p}, f)\}_{\mathfrak{p} \in S})$, where $S \subseteq P$ with S^c is finite,*
5. *$\overline{C(\mathfrak{p}, f)} = \Psi^*(\mathfrak{p})^{-1} C(\mathfrak{p}, f)$, for all $\mathfrak{p} \in P$ with $(\mathfrak{p}, \mathfrak{n}) = 1$.*

2.0.2. *Galois representations attached to f.* Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form of weight k, level $\mathfrak{n}$ and character Ψ with coefficient field E_f. For $\lambda \in \mathcal{P}_f$, by the works of Ohta, Carayol, Blasius-Rogawski and Taylor (cf. [Tay89] for more details), there exists a continuous Galois representation

$$\rho_{f,\lambda} : G_F \to \mathrm{GL}_2(E_{f,\lambda}),$$

which is absolutely irreducible, totally odd, unramified outside $\mathfrak{n}q$, where $q \in \mathbb{P}$ is the rational prime lying below λ and $E_{f,\lambda}$ is the completion of E_f at λ. The representation $\rho_{f,\lambda}$ has the following properties. For all primes $\mathfrak{p}$ of $\mathcal{O}_F$ with $(\mathfrak{p}, \mathfrak{n}q) = 1$, we have

$$\mathrm{tr}(\rho_{f,\lambda}(\mathrm{Frob}_{\mathfrak{p}})) = C(\mathfrak{p}, f) \text{ and } \det(\rho_{f,\lambda}(\mathrm{Frob}_{\mathfrak{p}})) = \Psi^*(\mathfrak{p}) N(\mathfrak{p})^{k_0 - 1} \tag{2.1}$$

(cf. [Car86]). By taking a Galois stable lattice, we define

$$\bar{\rho}_{f,\lambda} := \rho_{f,\lambda} \pmod{\lambda} : G_F \to \mathrm{GL}_2(\mathbb{F}_\lambda) \tag{2.2}$$

whose semi-simplification is independent of the particular choice of a lattice. We conclude this section by recalling the Chebotarev density theorem (cf. [Ser81]).

Theorem 2.2. *Let C be a conjugacy class of $G =: \bar{\rho}_{f,\lambda}(G_F)$. The natural density of $\{\mathfrak{p} \in P : [\bar{\rho}_{f,\lambda}(\mathrm{Frob}_{\mathfrak{p}})]_G = C\}$ is $\frac{|C|}{|G|}$.*

2.0.3. *Inner twists and its properties.* In this section, we define inner twists associated with a Hilbert modular form and describe some of their properties. This notion is quite useful in §4.

Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form defined over F, of weight k with Hecke character Ψ. For any Hecke character Φ of F, let f_Φ denote the twist of f by Φ (cf. [SW93, §5]). The Fourier coefficients of f and f_Φ are related as follows.

Proposition 2.3. [SW93, Proposition 5.1] *Let f and f_Φ be as above. If $\mathfrak{n}_0$ and $\mathfrak{m}_0$ are the conductors of Ψ and Φ, respectively, then $f_\Phi \in S_k\big(\mathrm{lcm}(\mathfrak{n}, \mathfrak{m}_0\mathfrak{n}_0, \mathfrak{m}_0^2), \Psi\Phi^2\big)$ and*

$$C(\mathfrak{m}, f_\Phi) = \Phi^*(\mathfrak{m}) C(\mathfrak{m}, f),$$

for all ideals $\mathfrak{m}$ of $\mathcal{O}_F$.

Definition 2.4. *We say a primitive form f is non-CM if there exists a non-trivial Hecke character Φ of F such that $C(\mathfrak{p}, f) = \Phi^*(\mathfrak{p}) C(\mathfrak{p}, f)$ for almost all prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$.*

We are now ready to define inner twists.

Definition 2.5 (Inner twists). *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a non-CM primitive form over F of weight k, level $\mathfrak{n}$ and character Ψ. For any Hecke character Φ of F, we say the twist f_Φ of f is inner if there exists a field automorphism $\gamma : E_f \to E_f$ such that $\gamma(C(\mathfrak{p}, f)) = C(\mathfrak{p}, f_\Phi)$ for almost all prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$.*

Remark 2.6. *For a non-CM form f, the identity map* $\mathrm{id} : E_f \to E_f$ *induces an inner twist of f and we refer it as the trivial inner twist of f.*

Let $\Gamma \subseteq \mathrm{Aut}(E_f)$ denote the set of all γ associated to all the inner twists of f. Similar to the classical case, we get that Γ is a subgroup of $\mathrm{Aut}(E_f)$ and $F_f = E_f^{\Gamma}$, where E_f^{Γ} is the fixed field of E_f by Γ. By Galois theory, E_f is a finite Galois extension of F_f. Some of the properties of F_f are given below.

Lemma 2.7. *The field F_f is totally real and $C^*(\mathfrak{p}, f) \in \mathbb{Q}(C(\mathfrak{p}, f))$.*

Proof. By Proposition 2.1, we have $C^*(\mathfrak{p}, f) = C(\mathfrak{p}, f)\overline{C(\mathfrak{p}, f)}$, which shows that F_f is totally real. By Proposition 2.1, if E_f is totally real, then $C^*(\mathfrak{p}, f) = C(\mathfrak{p}, f)^2 \in \mathbb{Q}(C(\mathfrak{p}, f))$. If E_f is a CM field, then $\mathbb{Q}(C(\mathfrak{p}, f))$ is preserved under complex conjugation. Hence, $C^*(\mathfrak{p}, f) = C(\mathfrak{p}, f)\overline{C(\mathfrak{p}, f)} \in \mathbb{Q}(C(\mathfrak{p}, f))$. □

We will now examine the existence of trivial and non-trivial inner twists for any primitive form f. More precisely,

Lemma 2.8. *If $f \in S_k(\mathfrak{n}, \Psi)$ is a non-CM primitive form over F with a non-trivial Hecke character Ψ, then f has a non-trivial inner twist.*

Proof. Let $\sigma : E_f \to E_f$ be an automorphism defined by $\sigma(x) = \overline{x}$, for all $x \in E_f$. By Proposition 2.1, we have $\sigma(C(\mathfrak{p}, f)) = \Psi^*(\mathfrak{p})^{-1} C(\mathfrak{p}, f)$ for all prime ideals $\mathfrak{p}$ with $(\mathfrak{p}, \mathfrak{n}) = 1$. By Proposition 2.3, f has a non-trivial inner twist given by $(\Psi)^{-1}$. □

We now give some examples of primitive forms with a non-trivial inner twist.

Example 1. *Consider a non-CM primitive form f defined over $F = \mathbb{Q}(\sqrt{2})$ of weight $(2, 2)$, level $[41,\ 41,\ 2\sqrt{2}-7]$ and with trivial character. This Hilbert modular form f is labelled as* `2.2.8.1-41.1-a` *in* [LMFDB]*. The coefficient field E_f of f is $\mathbb{Q}(\sqrt{2})$ and $F_f = \mathbb{Q}$.*

Example 2. *Consider a non-CM primitive form f defined over $F = \mathbb{Q}(\sqrt{3})$ of weight $(2, 2)$, level $[13,\ 13,\ \sqrt{3}+4]$ and with trivial character. This Hilbert modular form f is labelled as* `2.2.12.1-13.1-a` *in* [LMFDB]*. The coefficient field E_f of f is $\mathbb{Q}(\sqrt{2})$ and $F_f = \mathbb{Q}$.*

Example 3. *Consider a non-CM primitive form f defined over $F = \mathbb{Q}(\sqrt{6})$ of weight $(2, 2)$, level $[9,\ 3,\ 3]$ and with trivial character. This Hilbert modular form f is labelled as* `2.2.24.1-9.1-a` *in* [LMFDB]*. The coefficient field E_f of f is $\mathbb{Q}(\sqrt{6})$ and $F_f = \mathbb{Q}$.*

In Example 1, Example 2 and Example 3, the coefficient field $E_f \neq F_f$. Hence, these primitive forms f have a non-trivial inner twist.

Lemma 2.9. *Suppose $f \in S_k(\mathfrak{n}, \Psi)$ is a non-CM primitive form over F of weight k and character Ψ such that $[E_f : \mathbb{Q}]$ is an odd prime. If E_f is totally real, then f does not have any non-trivial inner twists. In particular, if $\Psi = \Psi_0$ is a trivial character, then E_f is totally real.*

Proof. Let $\mathfrak{p} \in P$ be a prime with $(\mathfrak{p}, \mathfrak{n}) = 1$. Since E_f is totally real, $C(\mathfrak{p}, f)^2 \in F_f$. Since $[E_f : \mathbb{Q}]$ is prime, the field F_f is either $\mathbb{Q}$ or E_f. If $F_f = \mathbb{Q}$, then $[\mathbb{Q}(C(\mathfrak{p}, f)) : \mathbb{Q}]$ is either 1 or 2. This contradicts to $[E_f : \mathbb{Q}]$ is odd prime. Therefore, $F_f = E_f$. Hence, f does not have any non-trivial inner twists. □

3. Statement and proof of the main theorem

In this section, we shall state and prove the main theorem of this article.

Theorem 3.1 (Main Theorem)**.** *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form defined over F of parallel weight 2, level $\mathfrak{n}$, and character Ψ, which is not a theta series. Further, assume that $[F : \mathbb{Q}] = [E_f : \mathbb{Q}]$ is an odd prime. Then*

$$\delta_D\left(\left\{\mathfrak{p} \in P : \mathbb{Q}\big(C(\mathfrak{p}, f)\big) = E_f\right\}\right) = 1,$$

where $\delta_D(S), E_f$ denote the Dirichlet density of $S \subseteq P$, the coefficient field of f, respectively.

3.1. **Images of the residual Galois representations:** We now determine the images of the λ-adic residual Galois representations attached to a primitive form of parallel weight 2. The work of Dimitrov in [Dim05] is quite influential in this section.

Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form defined over F of weight $k = (k_1, k_2, \ldots, k_n)$, level $\mathfrak{n}$ and character Ψ. Recall that, $k_0 = \max\{k_1, \ldots, k_n\}$ and ω be the mod-q cyclotomic character. Then the function $\bar{\Psi}\omega^{k_0-2}$ is a character on G_F. Let $\hat{F}$ be the compositum of the Galois closure of F in $\bar{\mathbb{Q}}$ and the subfield of $\bar{\mathbb{Q}}$ given by $(\bar{F})^{\mathrm{Ker}(\bar{\Psi}\omega^{k_0-2})}$. Then $\hat{F}$ is a Galois extension of F and $G_{\hat{F}} \trianglelefteq G_F$. In [Dim05, Proposition 3.9], he described the image $\bar{\rho}_{f,\lambda}(G_{\hat{F}})$ for almost all $q \in \mathbb{P}$. More precisely, he proved:

Proposition 3.2. *Let f be a primitive form which is not a theta series. Then there exists a power $\hat{q}$ of q such that for almost all $q \in \mathbb{P}$, we have either*

$$\bar{\rho}_{f,\lambda}(G_{\hat{F}}) \simeq \left\{g \in \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(g) \in (\mathbb{F}_q^\times)^{k_0-1}\right\},$$

or

$$\bar{\rho}_{f,\lambda}(G_{\hat{F}}) \simeq \left\{g \in \mathbb{F}_{\hat{q}^2}^\times \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(g) \in (\mathbb{F}_q^\times)^{k_0-1}\right\}.$$

3.2. **Key proposition in the proof of Theorem 3.1:** We will now determine the image of $\bar{\rho}_{f,\lambda}$ for primitive forms f as in Theorem 3.1. More precisely,

Proposition 3.3. *Let $f \in S_k(\mathfrak{n}, \Psi)$ be as in Theorem 3.1. For any $\lambda \subseteq E_f$ be a prime ideal lying above q, we have*

$$\bar{\rho}_{f,\lambda}(G_F) \simeq \{\gamma \in \mathrm{GL}_2(\mathbb{F}_{q^d}) : \det(\gamma) \in \mathbb{F}_q^\times\},$$

for infinitely many $q \in \mathbb{P}$ with $\mathrm{f}(\lambda/q) = d$.

Before we start the proof of Proposition 3.3, we recall some necessary results.

Proposition 3.4 ([Mar77])**.** *Let $K/\mathbb{Q}$ be a cyclic Galois extension of degree n. For $1 \leq r \mid n$, let $S_r := \{q \in \mathbb{P} : \mathrm{e}(\lambda|q) = 1 \ \& \ \mathrm{f}(\lambda/q) = r$ for some prime ideal $\lambda|q\}$. Then $\delta_D(S_r) = \frac{\varphi(r)}{n}$.*

Corollary 3.5. *Let f be as in Theorem 3.1. Then, there exists infinitely many primes $q \in \mathbb{P}$ which are inert in both F, E_f.*

For $q \in \mathbb{P}$, let λ, υ be prime ideals of E_f, F lying above q, respectively. Let I_υ be the inertia group at υ. We now give the proof of Proposition 3.3.

Proof. The proof of the proposition is similar to that of the technique in [DD06, Proposition 3.1]. In our case, $k_0 = 2$, $\Psi = \Psi_{\text{triv}}$ and hence $G_{\hat{F}} = G_F$. By Proposition 3.2, for all primes $q \gg 1$, there exists a power $\hat{q}$ of q, we have either $\bar{\rho}_{f,\lambda}(G_F) \simeq \{g \in \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(g) \in \mathbb{F}_q^\times\}$, or $\bar{\rho}_{f,\lambda}(G_F) \simeq \{g \in \mathbb{F}_{\hat{q}^2}^\times \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(g) \in \mathbb{F}_q^\times\}$. We now show that the later possibility will not occur.

Suppose $\bar{\rho}_{f,\lambda}(G_F) \simeq \{\gamma \in \mathbb{F}_{\hat{q}^2}^\times \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(\gamma) \in \mathbb{F}_q^\times\}$ for some prime power $\hat{q}$ of q with $q \gg 1$. Now, we argue as in the proof of [Dim05, Proposition 3.9], we get that $\mathbb{F}_q \subseteq \mathbb{F}_{\hat{q}^2} \subseteq \mathbb{F}_\lambda$. However, this cannot happen because d is odd and $2 | [\mathbb{F}_\lambda : \mathbb{F}_q]$. Therefore,

$$\bar{\rho}_{f,\lambda}(G_F) \simeq \{g \in \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(g) \in \mathbb{F}_q^\times\}$$

for $q \in \mathbb{P}$ with $q \gg 1$. By [DD06, Proposition 1], the possible fundamental characters for $\bar{\rho}_{f,\lambda}|I_\upsilon$ are of level d or $2d$ if $\mathrm{f}(\upsilon|q) = d$. Hence, we have $\mathbb{F}_{q^d} \subseteq \mathbb{F}_{\hat{q}} \subseteq \mathbb{F}_\lambda$. Now, choose a prime $q \in \mathbb{P}$ which is inert in both F and E_f. By Corollary 3.5, there exists infinitely many such primes. Since $\mathrm{f}(\lambda|q) = d$, the fundamental characters of level $2d$ cannot occur in $\bar{\rho}_{f,\lambda}|I_\upsilon$, therefore $\mathbb{F}_{q^d} = \mathbb{F}_{\hat{q}} = \mathbb{F}_\lambda$. Therefore, we have

$$\bar{\rho}_{f,\lambda}(G_F) \simeq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^d}) : \det(g) \in \mathbb{F}_q^\times\}$$

for infinitely many $q \in \mathbb{P}$ with $\mathrm{f}(\lambda/q) = \mathrm{f}(\upsilon/q) = d$. We are done with the proof. □

3.3. **A result for Hilbert modular forms of weight k:** The aim of this section is to prove an analogue of Theorem 3.1 for Hilbert modular forms of weight k. When k is of parallel weight 2 and $[F : \mathbb{Q}] = [E_f : \mathbb{Q}]$ is an odd prime, we show that the assumption in Theorem 3.6 is satisfied and hence we use it to prove Theorem 3.1.

Theorem 3.6. *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form defined over F of weight k, level $\mathfrak{n}$ and character Ψ. For any subfield $\mathbb{Q} \subseteq L \subsetneq E_f$, assume that*

$$\bar{\rho}_{f,\lambda}(G_{\hat{F}}) \supseteq \{\gamma \in \mathrm{GL}_2(\mathbb{F}_{q^{\mathrm{f}}}) : \det(\gamma) \in (\mathbb{F}_q^\times)^{k_0-1}\} \text{ with } \mathrm{f} = \mathrm{f}(\lambda|q), \tag{3.1}$$

for infinitely many $\lambda \in \mathcal{P}_f$ with $\mathrm{f}_{\lambda,E_f,L} > 1$, where $q \in \mathbb{P}$ lying below λ. Then

$$\delta_D\left(\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in L\}\right) = 0.$$

The following proposition of Koo et al., (cf. [KSW08, Proposition 2.1(c)]) is helpful in the proof of Theorem 3.6.

Proposition 3.7. *Let $R \subseteq \tilde{R}$ be two subgroups of $\mathbb{F}_{q^r}^\times$ for some $q \in \mathbb{P}$ and $r \in \mathbb{N}$. Let $G \subseteq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^r}) : \det(g) \in \tilde{R}\} \leq \mathrm{GL}_2(\mathbb{F}_{q^r})$. Let $P(x) = x^2 - ax + b \in \mathbb{F}_{q^r}[x]$. Then $\sum_C |C| \leq 2|\tilde{R}/R|(q^2+q)$, where the sum carries over all the conjugacy classes C of G with characteristic polynomial equals to $P(x)$.*

We now start the proof of Theorem 3.6.

Proof. Let $\mathcal{O}_{E_f}, \mathcal{O}_L$ denote the ring of integers of E_f, L, respectively. Let T be the set of all prime ideals λ of $\mathcal{O}_{E_f}$ such that (3.1) holds. By assumption, T is an infinite set. For any $Q \in T$, let Q_L, q be the prime ideals of $\mathcal{O}_L, \mathbb{Z}$ lying below Q, respectively. Let $\mathbb{F}_{q^r} = \mathcal{O}_L/Q_L,\ \mathbb{F}_{q^{rm}} = \mathcal{O}_{E_f}/Q$ for some $r \geq 1, m \geq 2$.

Let $R := (\mathbb{F}_q^\times)^{k_0-1}$, $W \leq \mathbb{F}_{q^{rm}}^\times$ denote the image of Ψ^* mod Q and $\tilde{R} := \langle R, W\rangle$, the subgroup of $\mathbb{F}_{q^{rm}}^\times$ generated by R and W. Then $|R| \leq q-1,\ |W| \leq |(\mathcal{O}_E/\mathfrak{n})^\times|$ and $|\tilde{R}| \leq |R||W|$.

Let $G := \bar{\rho}_{f,Q}(G_F)$ be the image of Q-adic residual Galois representation $\bar{\rho}_{f,Q}$. By (2.1) and (2.2), G is a subgroup of $\{g \in \mathrm{GL}_2(\mathbb{F}_{q^{rm}}) : \det(g) \in \tilde{R}\}$.

Let $M_Q := \bigsqcup_C \{\mathfrak{p} \in P : [\bar{\rho}_{f,Q}(\mathrm{Frob}_{\mathfrak{p}})]_G = C\}$, where C carries over all the conjugacy classes of G with characteristic polynomial $x^2 - ax + b \in \mathbb{F}_{q^{rm}}[x]$ such that $a \in \mathbb{F}_{q^r}$ and $b \in \tilde{R}$. Then, there are at most $q^r|R||W|$ such polynomials. By (2.1), we have $a \equiv C(\mathfrak{p}, f) \pmod{Q}$. Since $a \in \mathbb{F}_{q^r}$, we get $C(\mathfrak{p}, f) \pmod{Q} \in \mathbb{F}_{q^r}$. Hence,

$$M_Q \supseteq \{\mathfrak{p} \in P : C(\mathfrak{p}, f) \pmod{Q} \in \mathbb{F}_{q^r}\}. \tag{3.2}$$

By Theorem 2.2, we have $\delta_D(M_Q) = \sum_C \frac{|C|}{|G|}$. Now, by Proposition 3.7, we get

$$\delta_D(M_Q) \le \frac{q^r|R||W| \times 2|\tilde{R}/R|(q^2+q)}{|G|} = \frac{2|R||W|^2 q^r (q^2+q)}{|G|}. \tag{3.3}$$

Since $G_F \supseteq G_{\hat{F}}$, by (3.1), we get

$$|G| \ge |R| \times |\mathrm{SL}_2(\mathbb{F}_{q^{rm}})|, \tag{3.4}$$

and this provides a lower bound to $|G|$. Combining (3.4) with (3.3), we get

$$\delta_D(M_Q) \le \frac{2|W|^2|R|q^r(q^2+q)}{|R| \times |\mathrm{SL}_2(\mathbb{F}_{q^{rm}})|} = \frac{2|W|^2 q^{r+3}}{q^{3rm}(q-1)}.$$

Since $m \ge 2, r \ge 1$, we have $\delta_D(M_Q) \le O\left(\frac{1}{q^2}\right)$. By hypothesis, T is an infinite set and hence q is unbounded. The inclusion of the sets in (3.2) implies $\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in L\} \subseteq \bigcap_{Q \in T} M_Q$. Therefore, we have

$$\delta_D\left(\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in L\}\right) = 0. \tag{3.5}$$

This completes the proof of an auxiliary result, i.e., Theorem 3.6. ☐

The above theorem holds even if the inclusion in (3.1) holds up to conjugation.

Corollary 3.8. *Let f be as in Theorem 3.6 which satisfies* (3.1)*, for any subfield $\mathbb{Q} \subseteq L \subsetneq E_f$. Then $\delta_D\left(\{\mathfrak{p} \in P : \mathbb{Q}(C(\mathfrak{p}, f)) = E_f\}\right) = 1$.*

Proof. Let $\mathfrak{p} \in P$ be a prime with $\mathbb{Q}(C(\mathfrak{p}, f)) \subsetneq E_f$.Then $C(\mathfrak{p}, f) \in L$ for some proper subfield L of E_f. Since $[E_f : \mathbb{Q}]$ is a finite separable extension, there are only finitely many subfields between $\mathbb{Q}$ and E_f, and by Theorem 3.6, we have that $\delta_D\left(\{\mathfrak{p} \in P : \mathbb{Q}(C(\mathfrak{p}, f)) \subsetneq E_f\}\right) = 0$. This completes the proof of the corollary. ☐

We have some remarks to make.

- The conclusion of Theorem 3.6 implies that f has to be a non-CM form; For a CM form, the density of $\mathfrak{p} \in P$ for which $C(\mathfrak{p}, f) = 0$ has density 1/2.
- The equation (3.1) of Theorem 3.6 implies that $E_f \neq \mathbb{Q}$. However, if $E_f = \mathbb{Q}$ then $C(\mathfrak{p}, f) \in \mathbb{Q}$ for all $\mathfrak{p} \in P$. The density of $\{\mathfrak{p} \in P : \mathbb{Q}(C(\mathfrak{p}, f)) = \mathbb{Q}\}$ is 1 if f is non-CM, is 1/2 if f is CM.

3.4. **The proof of Theorem 3.1 with supporting examples.** In this section, we give a proof of Theorem 3.1 and provide some examples of f in support of it.

Proof of Theorem 3.1. Since $[E_f : \mathbb{Q}]$ is an odd prime, there only proper subfield L of E_f is $\mathbb{Q}$. By Proposition 3.3, f satisfies the assumption (3.1) of Theorem 3.6. Hence, by Corollary 3.8, the proof of Theorem 3.1 follows. ☐

We now give some examples of primitive Hilbert modular forms f in support of Theorem 3.1.

Example 4. *Consider a non-CM primitive form f defined over $F = \mathbb{Q}(\zeta_7)^+$ with generator ω having minimal polynomial $x^3 - x^2 - 2x + 1$, with weight $(2, 2, 2)$, level $[167,\ 167,\ \omega^2 + \omega - 8]$ and with trivial character. This Hilbert modular form f is labelled as* `3.3.49.1-167.1-a` *in* [LMFDB]. *The coefficient field E_f of f is $\mathbb{Q}(\alpha)$, where α is a root of the irreducible polynomial $x^3 - x^2 - 4x - 1 \in \mathbb{Q}[x]$.*

Example 5. *Consider a non-CM primitive form f defined over $F = \mathbb{Q}(\zeta_9)^+$ with generator ω having minimal polynomial $x^3 - 3x - 1$, with weight $(2, 2, 2)$, level $[71,\ 71,\ \omega^2 + \omega - 7]$ and with trivial character. This Hilbert modular form f is labelled as* `3.3.81.1-71.1-a` *in* [LMFDB]. *The coefficient field E_f of f is $\mathbb{Q}(\beta)$, where β is a root of the irreducible polynomial $x^3 - x^2 - 4x + 3 \in \mathbb{Q}[x]$.*

Example 6. *Consider a non-CM primitive form f defined over $F = \mathbb{Q}(\zeta_7)^+$ with generator ω having minimal polynomial $x^3 - x^2 - 2x + 1$, with weight $(2, 2, 2)$, level $[239,\ 239,\ 6\omega^2 - 5\omega - 7]$ and with trivial character. This Hilbert modular form f is labelled as* `3.3.49.1-239.1-a` *in* [LMFDB]. *The coefficient field E_f of f is $\mathbb{Q}(\theta)$, where θ is a root of the irreducible polynomial $x^3 - 12x - 8 \in \mathbb{Q}[x]$.*

The primitive modular forms f in Example 4, Example 5, and Example 6 are of parallel weight 2 with $[F : \mathbb{Q}] = [E_f : \mathbb{Q}] = 3$ and hence they satisfies the hypothesis of Theorem 3.1. Moreover, E_f is totally real and hence by Lemma 2.9, these primitive forms f do not have any non-trivial inner twists.

3.5. **Computation of some Dirichlet density for F_f.** In this section, we shall state and prove a variant of Theorem 3.6 and Corollary 3.8 for F_f. In fact, we compute the Dirichlet density of the set $\{\mathfrak{p} \in P : \mathbb{Q}(C^*(\mathfrak{p}, f)) = F_f\}$.

Theorem 3.9. *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form defined over F of weight k, level $\mathfrak{n}$ and character Ψ. For any subfield $\mathbb{Q} \subseteq L \subsetneq F_f$, assume that*

$$\bar{\rho}_{f,\lambda}(G_{\hat{F}}) \supseteq \{\gamma \in \mathrm{GL}_2(\mathbb{F}_{q^{\mathrm{f}}}) : \det(\gamma) \in (\mathbb{F}_q^\times)^{k_0 - 1}\} \text{ with } \mathrm{f} = \mathrm{f}_{\lambda, F_f, \mathbb{Q}}, \tag{3.6}$$

for infinitely many $\lambda \in \mathcal{P}_f$ with $\mathrm{f}_{\lambda, F_f, L} > 1$, where $q \in \mathbb{P}$ is the rational prime lying below λ. Then

$$\delta_D\left(\{\mathfrak{p} \in P :\ C^*(\mathfrak{p}, f) \in L\}\right) = 0.$$

The above theorem holds even if the inclusion in (3.6) holds up to conjugation.

Proof. In this proof, we follow the notations as in Theorem 3.6. Let $\mathcal{O}_{F_f}$ be the ring of integers of F_f. For any $Q \in T$, let Q_F be the prime ideal of $\mathcal{O}_{F_f}$ lying below Q. Let $\mathcal{O}_L/Q_L = \mathbb{F}_{q^r}, \mathcal{O}_{F_f}/Q_F = \mathbb{F}_{q^{rm}}$ and $\mathcal{O}_{E_f}/Q = \mathbb{F}_{q^{rms}}$, for some $r \geq 1,\ m \geq 2$ and $s \geq 1$. Then $G \subseteq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^{rms}}) : \det(g) \in \tilde{R}\}$. Now, arguing as in the proof of Theorem 3.6, we get $\delta_D(M_Q) \leq \frac{4|W|^3 q^{r+3}}{q^{3rm}(q-1)}$. Since $m \geq 2, r \geq 1$, we get $\delta_D(M_Q) \leq O\left(\frac{1}{q^2}\right)$. Therefore, we have

$$\delta_D\left(\{\mathfrak{p} \in P :\ C^*(\mathfrak{p}, f) \in L\}\right) = 0.$$

This completes the proof of Theorem 3.9. □

Corollary 3.10. *Let f be as in Theorem 3.9 which satisfies* (3.6), *for any subfield $\mathbb{Q} \subseteq L \subsetneq F_f$. Then*

$$\delta_D\left(\{\mathfrak{p} \in P : \mathbb{Q}\big(C^*(\mathfrak{p}, f)\big) = F_f\}\right) = 1.$$

Proof. Suppose $\mathfrak{p} \in P$ is a prime such that $L = \mathbb{Q}\big(C^*(\mathfrak{p}, f)\big) \subsetneq F_f$ is a proper subfield of F_f. Since $[F_f : \mathbb{Q}]$ is a finite separable extension, there are only finitely many subfields between $\mathbb{Q}$ and F_f, and by Theorem 3.9, we get

$$\delta_D\left(\left\{\mathfrak{p} \in P : \mathbb{Q}\big(C^*(\mathfrak{p}, f)\big) \subsetneq F_f\right\}\right) = 0.$$

This completes the proof of the corollary. □

Corollary 3.11. *Let f and $\bar{\rho}_{f,\lambda}$ be as in Theorem 3.9. Then, we have*

$$\delta_D\left(\left\{\mathfrak{p} \in P : F_f \subseteq \mathbb{Q}(C(\mathfrak{p}, f))\right\}\right) = 1.$$

Proof. Suppose $\mathfrak{p} \in P$ with $\mathbb{Q}(C^*(\mathfrak{p}, f)) = F_f$. From Lemma 2.7, we have $F_f = \mathbb{Q}(C^*(\mathfrak{p}, f)) \subseteq \mathbb{Q}(C(\mathfrak{p}, f))$. Corollary 3.10 implies the result. □

In Example 4, Example 5, and Example 6, we have E_f and F are of degree 3 over $\mathbb{Q}$ and $E_f = F_f$. Since there are no proper subfields of F_f, by Proposition 3.3, we conclude that these examples satisfy the hypothesis (3.6) of Theorem 3.9.

4. Computation of the Dirichlet density for subfields of E_f

In §3, we have computed the Dirichlet density of $\mathfrak{p} \in P$ such that $C(\mathfrak{p}, f)$ (resp., $C^*(\mathfrak{p}, f)$) generates E_f (resp., F_f). In this section, for any subfield K of E_f, we compute the Dirichlet density of the set $\left\{\mathfrak{p} \in P : \mathbb{Q}\big(C(\mathfrak{p}, f)\big) = K\right\}$. It is quite surprising to see that this density depends on whether $K \supseteq F_f$ or not.

We now calculate the density of $\mathfrak{p} \in P$ such that $C(\mathfrak{p}, f) \in K$ when $F_f \nsubseteq K$. The following lemma is an analog of [KSW08, Corollary 1.3(a)] for classical modular forms.

Lemma 4.1. *Let f be as in Theorem 3.9. Let $K \subseteq E_f$ be a subfield such that $F_f \nsubseteq K$. Then $\delta_D\left(\left\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in K\right\}\right) = \delta_D\left(\left\{\mathfrak{p} \in P : \mathbb{Q}\big(C(\mathfrak{p}, f)\big) = K\right\}\right) = 0.$*

Proof. Since $F_f \nsubseteq K$, we get $\left\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in K\right\} \subseteq \left\{\mathfrak{p} \in P : F_f \nsubseteq \mathbb{Q}(C(\mathfrak{p}, f))\right\}$. The proof now follows from Corollary 3.11. □

Let $\Gamma' = \{\gamma_1, \ldots, \gamma_r\} \leq \Gamma$ be a subgroup associated with all the inner twists of f and let $\Psi_{\gamma_1}, \ldots, \Psi_{\gamma_r}$ be the corresponding Hecke characters. Hence, the ideal Hecke characters $\Psi^*_{\gamma_1}, \ldots, \Psi^*_{\gamma_r}$ can be thought of as characters on G_F. For each $i \in \{1, 2, \ldots, r\}$, define $H_{\gamma_i} := \mathrm{Ker}(\Psi^*_{\gamma_i})$ and set $H^{\Gamma'} := \bigcap_{i=1}^r H_{\gamma_i} \leq G_F$. Let $K_{H^{\Gamma'}}$ denote the fixed field of $H^{\Gamma'}$. Hence, $F \subseteq K_{H^{\Gamma'}} \subseteq \bar{F}$.

Lemma 4.2. *Let $\Gamma', H^{\Gamma'}$ and $K_{H^{\Gamma'}}$ be as above. Then*

$$\{\mathfrak{p} \in P : \mathfrak{p} \text{ splits completely in } K_{H^{\Gamma'}}\} = \{\mathfrak{p} \in P : \Psi^*_{\gamma}(\mathfrak{p}) = 1,\ \forall \gamma \in \Gamma'\}.$$

Proof. Let $m = [K_{H^{\Gamma'}} : F]$.

$$\begin{aligned}
&\{\mathfrak{p} \in P :\ \mathfrak{p} \text{ splits completely in } K_{H^{\Gamma'}}\}\\
&= \{\mathfrak{p} \in P :\ \mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_m \text{ for prime ideals } \mathfrak{p}_j \text{ in } K_{H^{\Gamma'}} \text{ with } 1 \leq j \leq m\}\\
&= \{\mathfrak{p} \in P :\ \Psi^*_{\gamma_i}(\mathfrak{p}_j) = 1\ \forall i \in \{1, 2, \ldots, r\},\ \forall j \in \{1, 2, \ldots, m\}\}\\
&= \{\mathfrak{p} \in P :\ \Psi^*_{\gamma_i}(\mathfrak{p}) = 1\ \forall i \in \{1, 2, \ldots, r\}\}.
\end{aligned}$$

This completes the proof of the lemma. □

We are now in a position to compute the density of the set $\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in K\}$ if $F_f \subseteq K$. The following proposition generalizes [KSW08, Corollary 1.3(b)] to primitive Hilbert modular forms.

Proposition 4.3. *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a non-CM primitive form defined over F of weight k and character Ψ. For any subfield K with $F_f \subseteq K \subseteq E_f$, there exists a subgroup Γ' of Γ such that $K = E_f^{\Gamma'}$ and $\delta_D\left(\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in K\}\right) = \frac{1}{[K_{H^{\Gamma'}}:F]}$.*

Proof. Since E_f/F_f is Galois, there exists $\Gamma' \subseteq \Gamma$ such that $K = E_f^{\Gamma'}$. Hence,

$$\begin{aligned}\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in K\} &= \{\mathfrak{p} \in P : \gamma(C(\mathfrak{p}, f)) = C(\mathfrak{p}, f) \text{ for all } \gamma \in \Gamma'\} \\ &= \{\mathfrak{p} \in P : \Psi_\gamma^*(\mathfrak{p}) C(\mathfrak{p}, f) = C(\mathfrak{p}, f) \text{ for all } \gamma \in \Gamma'\}.\end{aligned}$$

Since $\delta_D\left(\{\mathfrak{p} \in P : C(\mathfrak{p}, f) = 0\}\right) = 0$ (cf. [DK20, Theorem 4.4(1)]) and by Chebotarev density theorem, we have

$$\begin{aligned}&\delta_D\left(\{\mathfrak{p} \in P : \Psi_\gamma^*(\mathfrak{p}) C(\mathfrak{p}, f) = C(\mathfrak{p}, f) \text{ for all } \gamma \in \Gamma'\}\right) \\ &\quad = \delta_D\left(\{\mathfrak{p} \in P : \Psi_\gamma^*(\mathfrak{p}) = 1 \text{ for all } \gamma \in \Gamma'\}\right) \\ &\quad \underset{\text{Lemma } 4.2}{=} \delta_D\left(\{\mathfrak{p} \in P : \mathfrak{p} \text{ splits completely in } K_{H^{\Gamma'}}\}\right) = \frac{1}{[K_{H^{\Gamma'}} : F]}.\end{aligned}$$

This completes the proof. □

The following corollary is an application of Proposition 4.3 and an analog of [KSW08, Corollary 1.4] for classical modular forms.

Corollary 4.4. *Let f, K be as in Proposition 4.3 and $K = E_f^{\Gamma'}$ for $\Gamma' \leq \Gamma$. Then*

$$\begin{aligned}&\delta_D\left(\{\mathfrak{p} \in P : \mathbb{Q}(C(\mathfrak{p}, f)) = K\}\right) \\ &\quad = \delta_D\left(\{\mathfrak{p} \in P : \Psi_\gamma^*(\mathfrak{p}) = 1,\ \forall \gamma \in \Gamma' \text{ and } \Psi_\omega^*(\mathfrak{p}) \neq 1,\ \forall \omega \in \Gamma - \Gamma'\}\right).\end{aligned}$$

These results illustrate that the Dirichlet density of $\mathfrak{p} \in P$ such that $\mathbb{Q}(C(\mathfrak{p}, f)) = K$, with $F_f \subseteq K \subseteq E_f$, is determined by the inner twists of f associated with K.

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