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Nonlocal nonlinear finite element analysis of composite plates using TSDT

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Abstract

In this work, nonlocal nonlinear finite element analysis of laminated composite plates using Reddy's third-order shear deformation theory (TSDT) [1] and Eringen's nonlocality [2] is presented. The governing equations of third order shear deformation theory with the von Kármán strains are derived employing the Eringen's [2] stress-gradient constitutive model. The principle of virtual displacement is used to derive the weak forms, and the displacement finite element models are developed using the weak forms. Four-noded rectangular conforming element with 8 degrees of freedom per node has been used. The coefficients of stiffness matrix and tangent stiffness matrix are presented along with nonlocal force vector. The developed finite element model can be employed to capture the small scale deviations from local continuum models caused by material inhomogeneity and the inter atomic and inter molecular forces. Numerical examples are presented to illustrate the effects of nonlocality, anisotropy, and the von Kármán type nonlinearity on the bending behaviour of laminated composite plates.

Keywords: Nonlocality; TSDT; nonlinearity; laminated composites; finite element analysis

Introduction

Idealization of materials as a continuum has serious limitations in analyzing the material at nano scale. This is mainly due to the fact that the classical continuum models do not account for internal material length scales. Every material possesses inhomogeneity at smaller length scale which induces a strongly nonlinear behaviour and local weakness of the material, which causes material instability and triggers strain localization. Experimental evidence suggests that at nano scale the conventional continuum description is not adequate to explain the response observed in experiments. On the other hand, atomic and molecular models are computationally

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expensive [3]. Several researchers [4], [5], [6] have tried to address this issue. Improved formulations were proposed by considering elastic materials with long range cohesive forces, elastic media with micro structure, and continuum approaches derived from an atomic lattice theory by Edelen et al.[7]. Nonlocal continuum models attempt to extend the continuum mechanics approach to smaller length scales by introducing a length scale in the constitutive relations. These nonlocal constitutive relations are proven to make the singularity disappear at the crack tip (see [8] and [9]) . Nonlocal models can also capture the size effects observed in experimental and discrete simulations. Pisano et al. [10] provided solutions for two-dimensional elasticity problems using a nonlocal finite element model. These models also reported to achieve properly convergent solutions for localized damage [11]. Isaac et al. [12] reported the measurement of size effect on the nominal strength of notched specimens of fiber-reinforced composites. Golmakani et al. [13] analyzed, using nonlocal continuum mechanics, the orthotropic nano scale plates resting on a Pasternak foundation and subjected to a transverse load. Abdollahi et al. [14] for the first time introduced a boundary layer method based on Eringen's nonlocal integral model. Reddy [15] presented nonlocal and nonlinear governing equations for the beams and plates using classical and shear deformation theories. Jirásek [16] explained how the classical continuum theory can be enriched to deal with problems such as (i) dispersion of short elastic waves in heterogeneous or discrete media, (ii) size effects in microscale elastoplasticity, (ii) localization of strain and damage in quasi brittle structures. Jan et al. [17] developed finite element models for large deformation analysis of piezoelectric nano plates using nonlocal and gradient theories. Banafsheh et al. [18] using a three dimensional strong nonlocal elasticity presented a formulation to capture the size dependent behaviour of plate structures as a function of their thickness. Sarkar et al. [19] explored the physical meaning of length scale parameter that is used in the nonlocal constitutive relations. There has been considerable focus in the recent years towards the development of generalized continuum theories that account for the inherent micro structure in natural engineering materials [20], [21], [22]. Wang et al. [23] noticed through nonlocal Euler Bernouli beam theory and Timoshenko beam theory that the length scale effect is noticeable for nano structures in their static responses. Raghu et al. [24] provided analytical solutions for linear analysis of laminated composite plates using Reddy's TSDT [1]. Rahmani et al. [25] studied the surface effect on the buckling of nano wires embedded in Winkler–Pasternak elastic medium based on nonlocal theory. Shahrokh et al. [26] studied the impact response of rectangular plate based on nonlocal elasticity theory. Fatima et al. [27] presented the nonlocal zeroth order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation. Here they considered the effect of shear deformation in the axial displacements in terms of shear force instead of shear displacements. Recently, Giovanni et al. [28] presented the paradoxes associated with Eringen's nonlocal model for analysing nonlocal elastic nano beams.

Accurate analysis of laminated composite plates has gained the attention of many researchers due to increased utilization of these in various applications such as civil, mechanical, aero space, sports and in many other industries. This is owing to the fact that laminates provide greater flexibility in tailoring the structural behaviour by changing the stacking sequence. Laminated composites possess high stiffness to weight ratio, high corrosion resistance and temperature resistance compared to the conventional materials. However they also exhibit technical difficulties in understanding their structural behaviour. Due to the fact that the laminates possess high in plane modulus to transverse shear modulus ratio, shear deformations effects are prominent when the plate is subjected to transverse loads. On the other hand, the classical laminated plate theory (CLPT) does not account for the shear deformations [29]. The first-order shear deformation theory (FSDT) accounts for the shear deformations in a simple way that it needs shear correction factor [29]. The TSDT [1] not only predicts the parabolic variation of transverse shear strains and shear stresses but also avoids the need for a shear correction factor. Further more, the use of linearized theory may not predict the actual behaviour of the structure when the applied loads are large. Hence one has to resort to nonlinear analysis, at least accounting for the moderate rotations, to obtain a realistic behavior of the structure.

Chao et al. [30] developed displacement finite element model based on a shear deformation theory accounting for transverse shear in the sense of Reissner–Mindlin's thick plate theory and the von Kármán's moderate rotations. Phan et al.[31] developed displacement finite element model based on conforming element which has 7 degrees of freedom at each node. Gajbir singh et al. [32] used the FSDT to determine large deflection analysis of thick laminates considering equal interpolation for all the primary variables. Jinseok et al.([33], [34], [35]) presented analytical and finite element solutions for the analysis of functionally graded plates using couple stress based third-order shear deformation theory. Putcha et al.[36] developed mixed finite element models for the nonlinear analysis of laminated composites based on higher-order shear deformation theory. Recently Sadek et al. [37] obtained the solutions for analysis of composite laminated beams using meshless methods with a novel radial point interpolation technique. Nicholas et al. [38] for the free vibration analysis of laminated composite plates, examined the stability and accuracy of Fourier expansion based differential quadrature techniques such as harmonic differential quadrature, Fourier differential quadrature and improved Fourier expansion-based differential quadrature methods via the strong form finite elements.

The weak forms of the governing equations suggest that the finite element approximation of $(u, v, w, \phi_x, \phi_y)$ should be the Lagrange type and w be approximated using Hermite type functions with $\frac{\partial^2 w}{\partial x \partial y}$ as one of the degrees of freedom; here (u, v) denote the in-plane displacements, w is the transverse displacement, and (ϕ_x, ϕ_y) are the rotations of the transverse normal about the y and x axes, respectively. The main objective of the present paper is to obtain the nonlocal nonlinear response of the laminated composite plates using the TSDT with conforming elements that has 8 degrees of freedom at each node: $u, v, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x \partial y}, \phi_x, \phi_y$

Eringen's Nonlocal Model

According to Eringen [2], the stress at a point in a continuum body is a function of the strains at all neighbor points of the continuum. Hence the effects of small scale and atomic forces are considered as material parameters in the constitutive equation. Following experimental observations, Eringen [20] proposed a constitutive model that expresses the nonlocal stress tensor $\boldsymbol{\sigma}^{nl}$ at point x as

$$\boldsymbol{\sigma}^{nl}(\boldsymbol{x}) = \int K(|\boldsymbol{x}' - \boldsymbol{x}|, \tau)\boldsymbol{\sigma}(\boldsymbol{x}')\, dv' \tag{1}$$

where $\boldsymbol{\sigma}(\boldsymbol{x}')$ is the classical macroscopic stress tensor at point $\boldsymbol{x}'$ and $K(|\boldsymbol{x}'-\boldsymbol{x}|, \tau)$ is the Kernel function which is normalized over the volume of the body represents the nonlocal modulus and τ is the material constant that depends on the internal characteristic length (e.g., lattice parameter, granular distance) and external characteristic length (e.g., crack length). From equation (1) it can be seen that K has the units of (length)$^{-3}$.

The Kernel function has the following properties [39]:

The function attains it's maximum at $\boldsymbol{x} = \boldsymbol{x'}$ and attenuates with $|\boldsymbol{x'} - \boldsymbol{x}|$.

When $\tau \to 0$, K becomes Dirac delta function. This makes nonlocal elasticity breaks down to classical elasticity.

Kernel function K can be determined by matching the dispersion curves of plane waves with those of atomic lattice dynamics. For two dimensional case, it can be found to be

$$K(|\boldsymbol{x}|, \tau) = (\pi \tau l^2)^{-1} exp(-\boldsymbol{x}.\boldsymbol{x}/l^2\tau) \tag{2}$$

Furthermore, K is assumed to be a Green's function of a linear differential operator $\mathcal{L}$.

$$\mathcal{L}K(|\boldsymbol{x'} - \boldsymbol{x}|, \tau) = \delta(|\boldsymbol{x'} - \boldsymbol{x}|) \tag{3}$$

Applying Eq. (3) to Eq. (1), we obtain

$$\mathcal{L}\sigma_{ij}^{nl} = \sigma_{ij} \tag{4}$$

If $\mathcal{L}$ is differential operator with constant coefficients, then

$$(\mathcal{L}\sigma_{ij}^{nl})_{,k} = \mathcal{L}\sigma_{ij,k}^{nl} \tag{5}$$

Using the Eq. (5) the equilibrium equation of two-dimensional, linearly elastic, isotropic body can be written as

$$\sigma_{kl,k} + \mathcal{L}(\rho f_l - \rho \ddot{u}_l) = 0 \tag{6}$$

Equation (4) can be represented equivalently in differential form as

$$\left(1 - \tau^2 l^2 \nabla^2\right) \boldsymbol{\sigma}^{nl} = \boldsymbol{\sigma} \tag{7}$$

where $\tau = \frac{(e_0 a)^2}{l^2}$, e_0 is a material constant and a and l are the internal and external characteristic lengths, respectively (see [2]). In general, ∇^2 is the three-dimensional Laplace operator. The nonlocal parameter μ is defined as $\mu = \tau^2 l^2$. In the component form, the relation can be written as

$$\mathcal{L}(\sigma_{ij}^{nl}) = \sigma_{ij} = C_{ijmn}\varepsilon_{mn} \tag{8}$$

Third-order shear deformation theory (TSDT)

The classical beam or plate theories neglect transverse shear strains by making the assumption that transverse normals to the plane of the plate before bending remain normal after bending. Transverse shear strains cannot be neglected in analyzing structures which possess low value of shear moduli compared to the in-plane moduli, especially, in laminated composite plates and shells. The FSDT (see Mindlin [40] and Reddy [29]) predicts a constant variation of the transverse shear strain and, hence, constant transverse shear stresses through the laminate thickness. Although the actual variations of the transverse shear strains and stresses cannot be altered, the transverse shear forces can be corrected using shear correction factors.

The TSDT of Reddy [1] relaxes the assumptions made in the classical plate theory. In particular, in the TSDT, plane sections normal to the midplane before bending are not assumed to remain normal to the midsurface after bending. In fact, the sections can become surfaces. This assumption allows quadratic representation of the transverse shear strains through the plate thickness.

Displacement field

In the third-order shear deformation theory, the displacement field is expanded up to third degree of the thickness coordinate:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\phi_x - \frac{4z^3}{3h^2}\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \\ v(x,y,z) &= v_0(x,y) + z\phi_y - \frac{4z^3}{3h^2}\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{9}$$

where (u_0, v_0, w_0) are in-plane displacements of a point on the mid-plane (i.e., $z = 0$). ϕ_x and ϕ_y denote the rotations of a transverse normal line at the mid-plane ($\phi_x = \frac{\partial u}{\partial z}$ and $\phi_y = \frac{\partial v}{\partial z}$). The total thickness of the laminate is denoted by h. The cubic variation of the displacement field with the thickness coordinate allows a parabolic variation of the transverse shear strains and shear stresses and avoids the need for shear correction factors.

Strain–displacement relations

The Green–Lagrange strain components that account for the geometric nonlinearity in the third-order shear deformation theory are

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} \tag{10}$$

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + z^2 \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} \tag{11}$$

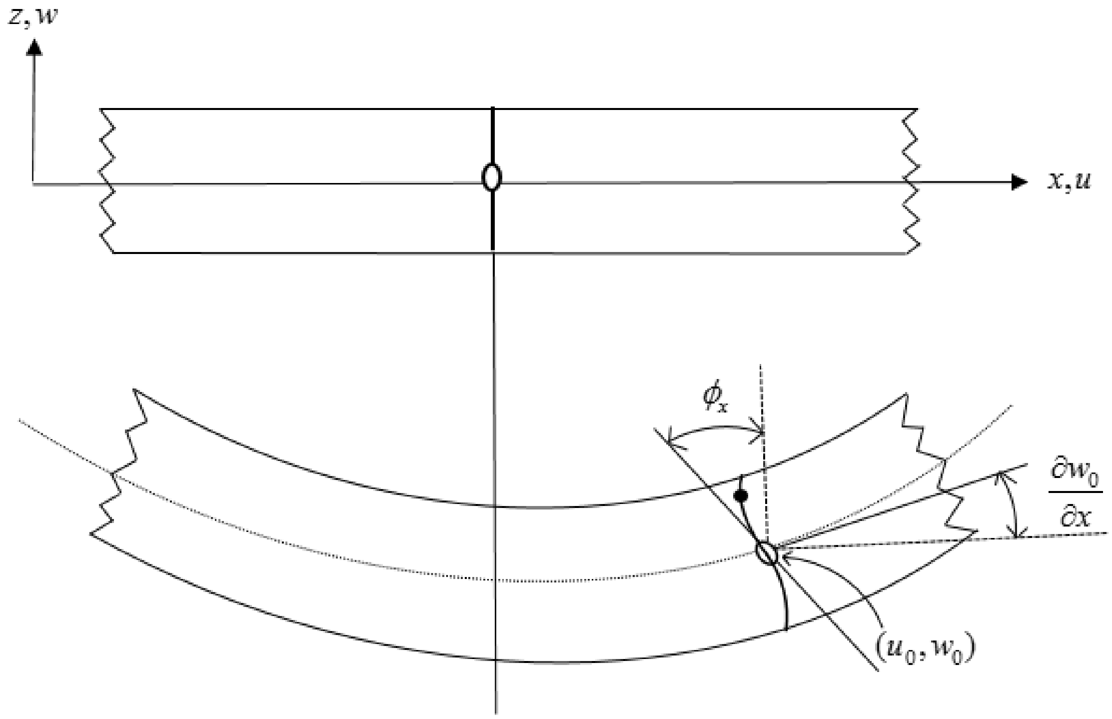

Figure 1: Deformation of transverse normal according to the third order plate theory

where

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}(\frac{\partial w_0}{\partial x})^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}(\frac{\partial w_0}{\partial y})^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \tag{12}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = -c_1 \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \tag{13}$$

$$\begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \phi_y + \frac{\partial w_0}{\partial y} \\ \phi_x + \frac{\partial w_0}{\partial x} \end{Bmatrix}, \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} = -c_2 \begin{Bmatrix} \phi_y + \frac{\partial w_0}{\partial y} \\ \phi_x + \frac{\partial w_0}{\partial x} \end{Bmatrix} \tag{14}$$

Lamina constitutive relations

Since the laminate is made of several orthotropic layers, with their material axes oriented arbitrarily with respect to laminate coordinates, the constitutive equations of each layer must be transformed to the laminate coordinates (x, y, z). The transformed stress–strain relations in the laminate coordinates (x, y, z) are given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{15}$$

where

$$\begin{aligned} \bar{Q}_{11} &= Q_{11}\cos^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta \\ \bar{Q}_{12} &= \left(Q_{11} + Q_{22} - 4Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{12}\left(\sin^4\theta + \cos^4\theta\right) \\ \bar{Q}_{16} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)\sin\theta\cos^3\theta + \left(Q_{12} - Q_{22} + 2Q_{66}\right)\sin^3\theta\cos\theta \\ \bar{Q}_{22} &= Q_{11}\sin^4\theta + 2\left(Q_{16} + 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta \\ \bar{Q}_{26} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)\sin^3\theta\cos\theta + \left(Q_{12} - Q_{22} + 2Q_{66}\right)\sin\theta\cos^3\theta \\ \bar{Q}_{66} &= \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta) \end{aligned} \tag{16}$$

$$\begin{aligned} \bar{Q}_{44} &= Q_{44}\cos^2\theta + Q_{55}\sin^2\theta \\ \bar{Q}_{45} &= \left(Q_{55} - Q_{44}\right)\cos\theta\sin\theta \\ \bar{Q}_{55} &= Q_{44}\sin^2\theta + Q_{55}\cos^2\theta \end{aligned} \tag{17}$$

where

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}, \tag{18}$$

$$Q_{16} = Q_{26} = 0, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13} \tag{19}$$

where θ is the orientation, measured in counterclockwise, from the fiber direction to the positive x-axis, E_1 and E_2 are elastic moduli, ν_{12} and ν_{21} are Poisson's ratios, and G_{12}, G_{13} and G_{23} are the shear moduli.

Governing equations for nonlocal TSDT

The governing equations for the TSDT are derived by using the dynamic version of the principle of virtual displacements. The statement of the principle of virtual work is given by

$$0 = \int_T (\delta U + \delta V - \delta K)dt \tag{20}$$

where δU is the virtual strain energy, δV is the virtual work done by applied forces, and δK is the virtual kinetic energy. Following the principle of virtual work, the equations of equilibrium are derived as follows:

$$\frac{\partial N_{xx}^{nl}}{\partial x} + \frac{\partial N_{xy}^{nl}}{\partial y} = 0 \tag{21}$$

$$\frac{\partial N_{xy}^{nl}}{\partial x} + \frac{\partial N_{yy}^{nl}}{\partial y} = 0 \tag{22}$$

$$\frac{\partial \bar{Q}_x^{nl}}{\partial x} + \frac{\partial \bar{Q}_y^{nl}}{\partial y} + \frac{\partial}{\partial x}\left(N_{xx}^{nl}\frac{\partial w_0}{\partial x} + N_{yy}^{nl}\frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}^{nl}\frac{\partial w_0}{\partial x} + N_{yy}^{nl}\frac{\partial w_0}{\partial y}\right) + c_1\left(\frac{\partial^2 P_{xx}^{nl}}{\partial x^2} + 2\frac{\partial^2 P_{xy}^{nl}}{\partial x\,\partial y} + \frac{\partial^2 P_{yy}^{nl}}{\partial y^2}\right) + q = 0 \tag{23}$$

$$\frac{\partial \bar{M}_{xx}^{nl}}{\partial x} + \frac{\partial \bar{M}_{xy}^{nl}}{\partial y} - \bar{Q}_x^{nl} = 0 \tag{24}$$

$$\frac{\partial \bar{M}_{xy}^{nl}}{\partial x} + \frac{\partial \bar{M}_{yy}^{nl}}{\partial y} - \bar{Q}_y^{nl} = 0 \tag{25}$$

where $c_1 = \frac{4}{3h^2}$ and

$$\bar{M}_{\alpha\beta}^{nl} = M_{\alpha\beta}^{nl} - c_1 P_{\alpha\beta}^{nl}, \quad \bar{Q}_{\alpha}^{nl} = Q_{\alpha}^{nl} - c_2 R_{\alpha}^{nl}, c_2 = 3c_1$$

The relations between the local and nonlocal stress resultants can be derived using eq. (8) as

$$\mathcal{L}(N_{\alpha\beta}^{nl}) = N_{\alpha\beta}, \quad \mathcal{L}(M_{\alpha\beta}^{nl}) = M_{\alpha\beta}, \quad \mathcal{L}(P_{\alpha\beta}^{nl}) = P_{\alpha\beta} \quad \mathcal{L}(Q_{\alpha}^{nl}) = Q_{\alpha}, \quad \mathcal{L}(R_{\alpha}^{nl}) = R_{\alpha} \tag{26}$$

$$\begin{Bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} dz, \begin{Bmatrix} Q_{\alpha} \\ R_{\alpha} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha z} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} dz \tag{27}$$

where α, β take the symbols x and y; q is the transverse load. The stress resultants in terms of strains can be written as follows:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} \tag{28}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} \tag{29}$$

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + \begin{bmatrix} H_{11} & H_{12} & H_{16} \\ H_{12} & H_{22} & H_{26} \\ H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{bmatrix} \tag{30}$$

$$\begin{Bmatrix} Q_{yz} \\ Q_{xz} \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} \tag{31}$$

$$\begin{Bmatrix} R_{yz} \\ R_{xz} \end{Bmatrix} = \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + \begin{bmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} \tag{32}$$

$$\{A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} \left(1, z, z^2, z^3, z^4, z^6\right) \, dz \quad (i, j = 1, 2, 6) \tag{33}$$

$$\{A_{ij}, D_{ij}, F_{ij}\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} \left(1, z^2, z^4\right) \, dz \quad (i, j = 4, 5) \tag{34}$$

To derive the governing equations of the nonlocal theory in terms of local stress resultants, we apply the operator $\mathcal{L}$ on both sides of the equations (21) - (25). Making use of the relations in the eq. (26), we obtain the following governing equations:

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{35}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0 \tag{36}$$

$$\frac{\partial \bar{Q}_x}{\partial x} + \frac{\partial \bar{Q}_y}{\partial y} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right) \tag{37}$$
$$+c_1\left(\frac{\partial^2 P_{xx}}{\partial x^2} + 2\frac{\partial^2 P_{xy}}{\partial x\,\partial y} + \frac{\partial^2 P_{yy}}{\partial y^2}\right) = -q\left(1 - \mu\nabla^2\right)$$

$$\frac{\partial \bar{M}_{xx}}{\partial x} + \frac{\partial \bar{M}_{xy}}{\partial y} - \bar{Q}_x = 0 \tag{38}$$
$$\frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_{yy}}{\partial y} - \bar{Q}_y = 0 \tag{39}$$

This completes the derivation of nonlocal governing equations. Note that the nonlocal boundary conditions remain same as the local boundary conditions.

Finite element model

A displacement finite element model of the governing equations is developed in this section. The weak forms for the equations (35) - (39) are obtained as follows:

$$0 = \int_{\Omega^e}\left[N_{xx}\delta u_{0,x} + N_{xy}\delta u_{0,y}\right]dxdy - \oint_{\Gamma^e}(\hat{n}_x N_{xx}\delta u_0 + \hat{n}_y N_{xy}\delta u_0)ds \tag{40}$$

$$0 = \int_{\Omega^e}\left[N_{xy}\delta v_{0,x} + N_{yy}\delta v_{0,y}\right]dxdy - \oint_{\Gamma^e}(\hat{n}_x N_{xy}\delta v_0 + \hat{n}_y N_{yy}\delta v_0)ds \tag{41}$$

$$\begin{aligned}
0 = \int_{\Omega^e}\Big\{&\bar{Q}_x\delta w_{0,x} + \bar{Q}_y\delta w_{0,y} + (N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y})\delta w_{0,x} + (N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y})\delta w_{0,y} \\
&- c_1(P_{xx}\delta w_{0,xx} + P_{yy}\delta w_{0,yy} + 2P_{xy}\delta w_{0,xy}) - [1 - \mu\nabla^2]q\delta w_0\Big\}dxdy \\
&- \oint_{\Gamma}\Big\{(\bar{Q}_x\hat{n}_x + \bar{Q}_y\hat{n}_y)\delta w_0 + (N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y})\hat{n}_x\delta w_0 + (N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y})\hat{n}_y\delta w_0 \\
&+ c_1\left[\frac{\partial P_{xx}}{\partial x}\hat{n}_x + \frac{\partial P_{yy}}{\partial y}\hat{n}_y + (\frac{\partial P_{xy}}{\partial x}\hat{n}_y + \frac{\partial P_{xy}}{\partial y}\hat{n}_x)\right]\delta w_0 ds \\
&- c_1\left[P_{xx}\frac{\partial \delta w_0}{\partial x}\hat{n}_x + P_{yy}\frac{\partial \delta w_0}{\partial y}\hat{n}_y + (P_{xy}\frac{\partial \delta w_0}{\partial x}\hat{n}_y + P_{xy}\frac{\partial \delta w_0}{\partial y}\hat{n}_x)\right]\Big\}ds
\end{aligned} \tag{42}$$

$$0 = \int_{\Omega^e}\left(\bar{Q}_x\delta\phi_x + \bar{M}_x\delta\phi_{x,x} + \bar{M}_{xy}\delta\phi_{x,y}\right)dxdy - \oint_{\Gamma^e}\left(M_{xx}\hat{n}_x\delta\phi_x + M_{xy}\hat{n}_y\delta\phi_y\right)ds \tag{43}$$

$$0 = \int_{\Omega^e}\left(\bar{Q}_y\delta\phi_y + \bar{M}_y\delta\phi_{y,y} + \bar{M}_{xy}\delta\phi_{y,x}\right)dxdy - \oint_{\Gamma^e}\left(M_{yy}\hat{n}_y\delta\phi_y + M_{xy}\hat{n}_x\delta\phi_x\right)ds \tag{44}$$

Finite Element Approximations

The primary variables of the Reddy's third-order theory are (as derived from the weak forms)

$$\left\{ u,\ v,\ w,\ \frac{\partial w}{\partial x},\ \frac{\partial w}{\partial y},\frac{\partial^2 w}{\partial x \partial y}\ \phi_x,\ \phi_y \right\}$$

Therefore, we must interpolate (u, v, ϕ_x, ϕ_y) using the Lagrange family of approximations, while w must be interpolated using the Hermite family of approximations (where the variable and its derivatives are interpolated). We seek the finite element approximations in the following form:

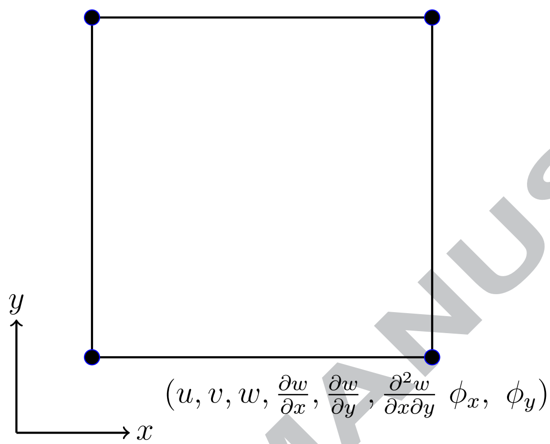

Figure 2: Four-noded rectangular element

$$u(x, y, t) \approx \sum_{j=1}^{m} U_j(t)\psi_j^{(1)}(x, y) \tag{45}$$

$$v(x, y, t) \approx \sum_{j=1}^{m} V_j(t)\psi_j^{(1)}(x, y) \tag{46}$$

$$w(x, y, t) \approx \sum_{j=1}^{n} \bar{\Delta}_j(t)\varphi_j(x, y) \tag{47}$$

$$\phi_x(x, y, t) \approx \sum_{j=1}^{n} \mathcal{X}_j(t)\psi_j^{(2)}(x, y) \tag{48}$$

$$\phi_y(x, y, t) \approx \sum_{j=1}^{n} \mathcal{Y}_j(t)\psi_j^{(2)}(x, y) \tag{49}$$

In this work, the same degree of interpolation for (u, v) and (ϕ_x, ϕ_y) is used. In general $\psi_j^{(1)} \neq \psi_j^{(2)}$. Here $\bar{\Delta}_j$ denote $(w, \partial w/\partial x, \partial w/\partial y, \partial^2 w/\partial x\partial y)$ at each node of the finite element (known as the *conforming* element).

Substitution of the approximations from Eqs. (45)–(49) into the weak forms in Eqs. (40)–(44)), we obtain the following finite element equations (for static bending analysis):

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} & \mathbf{K}^{14} & \mathbf{K}^{15} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} & \mathbf{K}^{24} & \mathbf{K}^{25} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} & \mathbf{K}^{34} & \mathbf{K}^{35} \\ \mathbf{K}^{41} & \mathbf{K}^{42} & \mathbf{K}^{43} & \mathbf{K}^{44} & \mathbf{K}^{45} \\ \mathbf{K}^{51} & \mathbf{K}^{52} & \mathbf{K}^{53} & \mathbf{K}^{54} & \mathbf{K}^{55} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \mathbf{V} \\ \bar{\mathbf{\Delta}} \\ \mathcal{X} \\ \mathcal{Y} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \\ \mathbf{F}^4 \\ \mathbf{F}^5 \end{Bmatrix}$$

$$\mathbf{K}\mathbf{\Delta} = \mathbf{F} \tag{50}$$

The coefficients of the stiffness matrix are as follows:

$$K_{ij}^{11} = \int_{\Omega^e} \left(A_{11} \frac{\partial \psi_j^{(1)}}{\partial x} \frac{\partial \psi_i^{(1)}}{\partial x} + A_{66} \frac{\partial \psi_j^{(1)}}{\partial y} \frac{\partial \psi_i^{(1)}}{\partial y} + A_{16} (\frac{\partial \psi_j^{(1)}}{\partial y} \frac{\partial \psi_i^{(1)}}{\partial x} + \frac{\partial \psi_j^{(1)}}{\partial x} \frac{\partial \psi_i^{(1)}}{\partial y}) \right) dxdy \tag{51}$$

$$K_{ij}^{12} = \int_{\Omega^e} \left(A_{12} \frac{\partial \psi_j^{(1)}}{\partial y} \frac{\partial \psi_i^{(1)}}{\partial x} + A_{66} \frac{\partial \psi_j^{(1)}}{\partial x} \frac{\partial \psi_i^{(1)}}{\partial y} + A_{16} \frac{\partial \psi_j^{(1)}}{\partial x} \frac{\partial \psi_i^{(1)}}{\partial x} + A_{26} \frac{\partial \psi_j^{(1)}}{\partial y} \frac{\partial \psi_i^{(1)}}{\partial y} \right) dxdy \tag{52}$$

$$\begin{aligned} K_{ij}^{13} = \int_{\Omega^e} \Bigg[\frac{\partial \psi_i^{(1)}}{\partial x} \Bigg(& \frac{1}{2} A_{11} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{12} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + \frac{1}{2} A_{16} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 E_{11} \frac{\partial^2 \varphi_j}{\partial x^2} \\ & - c_1 E_{12} \frac{\partial^2 \varphi_j}{\partial y^2} - 2 c_1 E_{16} \frac{\partial^2 \varphi_j}{\partial x \partial y} \Bigg) \\ & + \frac{\partial \psi_i^{(1)}}{\partial y} \Bigg(\frac{1}{2} A_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{26} \frac{\partial w}{\partial y} \frac{\partial \psi_j}{\partial y} + \frac{1}{2} A_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 E_{16} \frac{\partial^2 \varphi_j}{\partial x^2} \\ & - c_1 E_{26} \frac{\partial^2 \varphi_j}{\partial y^2} - 2 c_1 E_{66} \frac{\partial^2 \varphi_j}{\partial x \partial y} \Bigg) \Bigg] dxdy \end{aligned} \tag{53}$$

$$\begin{aligned} K_{ij}^{14} = \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(1)}}{\partial x} \left(B_{11} \frac{\partial \psi_j^{(2)}}{\partial x} + B_{16} \frac{\partial \psi_j^{(2)}}{\partial y} - c_1 E_{11} \frac{\partial \psi_j^{(2)}}{\partial x} - c_1 E_{16} \frac{\partial \psi_j^{(2)}}{\partial y} \right) \\ & + \frac{\partial \psi_i^{(1)}}{\partial y} \left(B_{16} \frac{\partial \psi_j^{(2)}}{\partial x} + B_{66} \frac{\partial \psi_j^{(2)}}{\partial y} - c_1 E_{16} \frac{\partial \psi_j^{(2)}}{\partial x} - c_1 E_{66} \frac{\partial \psi_j^{(2)}}{\partial y} \right) \Bigg] dxdy \end{aligned} \tag{54}$$

$$\begin{aligned} K_{ij}^{15} = \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(1)}}{\partial x} \left(B_{12} \frac{\partial \psi_j^{(2)}}{\partial y} + B_{16} \frac{\partial \psi_j^{(2)}}{\partial x} - c_1 E_{22} \frac{\partial \psi_j^{(2)}}{\partial y} - c_1 E_{26} \frac{\partial \psi_j^{(2)}}{\partial x} \right) \\ & + \frac{\partial \psi_i^{(1)}}{\partial y} \left(B_{26} \frac{\partial \psi_j^{(2)}}{\partial y} + B_{66} \frac{\partial \psi_j^{(2)}}{\partial x} - c_1 E_{26} \frac{\partial \psi_j^{(2)}}{\partial y} - c_1 E_{66} \frac{\partial \psi_j^{(2)}}{\partial x} \right) \Bigg] dxdy \end{aligned} \tag{55}$$

$$K_{ij}^{21} = \int_{\Omega^e} \left(A_{12} \frac{\partial \psi_j^{(1)}}{\partial x} \frac{\partial \psi_i^{(1)}}{\partial y} + A_{16} \frac{\partial \psi_j^{(1)}}{\partial x} \frac{\partial \psi_i^{(1)}}{\partial x} + A_{26} \frac{\partial \psi_j^{(1)}}{\partial y} \frac{\partial \psi_i^{(1)}}{\partial y} + A_{66} \frac{\partial \psi_j^{(1)}}{\partial y} \frac{\partial \psi_i^{(1)}}{\partial x} \right) dxdy \tag{56}$$

$$K_{ij}^{22} = \int_{\Omega^e} \left(A_{22} \frac{\partial \psi_j^{(1)}}{\partial y} \frac{\partial \psi_i^{(1)}}{\partial y} + A_{26} (\frac{\partial \psi_j^{(1)}}{\partial x} \frac{\partial \psi_i^{(1)}}{\partial y} + \frac{\partial \psi_j^{(1)}}{\partial y} \frac{\partial \psi_i^{(1)}}{\partial x}) + A_{66} \frac{\partial \psi_j^{(1)}}{\partial x} \frac{\partial \psi_i^{(1)}}{\partial x} \right) dxdy \tag{57}$$

$$\begin{aligned} K_{ij}^{23} = \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(1)}}{\partial y} \Bigg(\frac{1}{2} A_{12} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{22} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + \frac{1}{2} A_{26} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y}) - c_1 E_{12} \frac{\partial^2 \varphi_j}{\partial x^2} \\ & - c_1 E_{22} \frac{\partial^2 \varphi_j}{\partial y^2} - 2 c_1 E_{26} \frac{\partial^2 \varphi_j}{\partial x \partial y} \Bigg) \\ & + \frac{\partial \psi_i^{(1)}}{\partial x} \Bigg(\frac{1}{2} A_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{26} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + \frac{1}{2} A_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y}) - c_1 E_{16} \frac{\partial^2 \psi_j}{\partial x^2} \\ & - c_1 E_{26} \frac{\partial^2 \varphi_j}{\partial y^2} - 2 c_1 E_{66} \frac{\partial^2 \varphi_j}{\partial x \partial y} \Bigg) \Bigg] dxdy \end{aligned} \tag{58}$$

$$\begin{aligned} K_{ij}^{24} = \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(1)}}{\partial y} \Bigg(B_{21} \frac{\partial \psi_j^{(2)}}{\partial x} + B_{26} \frac{\partial \psi_j^{(2)}}{\partial y} - c_1 E_{12} \frac{\partial \psi_j^{(2)}}{\partial x} - c_1 E_{26} \frac{\partial \psi_j^{(2)}}{\partial y} \Bigg) \\ & + \frac{\partial \psi_i^{(1)}}{\partial x} \Bigg(B_{16} \frac{\partial \psi_j^{(2)}}{\partial x} + B_{66} \frac{\partial \psi_j^{(2)}}{\partial y} - c_1 E_{16} \frac{\partial \psi_j^{(2)}}{\partial x} - c_1 E_{66} \frac{\partial \psi_j^{(2)}}{\partial y} \Bigg) \Bigg] dxdy \end{aligned} \tag{59}$$

$$\begin{aligned} K_{ij}^{25} = \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(1)}}{\partial y} \Bigg(B_{22} \frac{\partial \psi_j^{(2)}}{\partial y} + B_{26} \frac{\partial \psi_j^{(2)}}{\partial x} - c_1 E_{22} \frac{\partial \psi_j^{(2)}}{\partial y} - c_1 E_{26} \frac{\partial \psi_j^{(2)}}{\partial x} \Bigg) \\ & + \frac{\partial \psi_i^{(1)}}{\partial x} \Bigg(B_{26} \frac{\partial \psi_j^{(2)}}{\partial y} + B_{66} \frac{\partial \psi_j^{(2)}}{\partial x} - c_1 E_{26} \frac{\partial \psi_j^{(2)}}{\partial y} - c_1 E_{66} \frac{\partial \psi_j^{(2)}}{\partial x} \Bigg) \Bigg] dxdy \end{aligned} \tag{60}$$

$$\begin{aligned} K_{ij}^{31} = \int_{\Omega^e} \Bigg[& \frac{\partial \varphi_i}{\partial x} \Bigg(A_{11} \frac{\partial w}{\partial x} \frac{\partial \psi_j^{(1)}}{\partial x} + A_{16} \frac{\partial w}{\partial x} \frac{\partial \psi_j^{(1)}}{\partial y} + A_{16} \frac{\partial w}{\partial y} \frac{\partial \psi_j^{(1)}}{\partial x} + A_{66} \frac{\partial w}{\partial y} \frac{\partial \psi_j^{(1)}}{\partial y} \Bigg) \\ & + \frac{\partial \varphi_i}{\partial y} \Bigg(A_{12} \frac{\partial w}{\partial y} \frac{\partial \psi_j^{(1)}}{\partial x} + A_{16} \frac{\partial w}{\partial x} \frac{\partial \psi_j^{(1)}}{\partial x} + A_{26} \frac{\partial w}{\partial y} \frac{\partial \psi_j^{(1)}}{\partial y} + A_{66} \frac{\partial w}{\partial x} \frac{\partial \psi_j^{(1)}}{\partial y} \Bigg) \\ & + \frac{\partial^2 \varphi_i}{\partial x^2} \Bigg(- c_1 E_{11} \frac{\partial \psi_j^{(1)}}{\partial x} - c_1 E_{16} \frac{\partial \psi_j^{(1)}}{\partial y} \Bigg) + \frac{\partial^2 \varphi_i}{\partial y^2} \Bigg(- c_1 E_{12} \frac{\partial \psi_j^{(1)}}{\partial x} - c_1 E_{26} \frac{\partial \psi_j^{(1)}}{\partial y} \Bigg) \\ & + \frac{\partial^2 \varphi_i}{\partial x \partial y} \Bigg(- 2 c_1 E_{16} \frac{\partial \psi_j^{(1)}}{\partial x} - 2 c_1 E_{66} \frac{\partial \psi_j^{(1)}}{\partial y} \Bigg) \Bigg] dxdy \end{aligned} \tag{61}$$

$$
\begin{aligned}
K_{ij}^{32} = \int_{\Omega^e} \Bigg[& \frac{\partial \varphi_i}{\partial x} \Bigg(A_{12} \frac{\partial w}{\partial x} \frac{\partial \psi_j^{(1)}}{\partial y} + A_{16} \frac{\partial w}{\partial x} \frac{\partial \psi_j^{(1)}}{\partial x} + A_{26} \frac{\partial w}{\partial y} \frac{\partial \psi_j^{(1)}}{\partial y} + A_{66} \frac{\partial w}{\partial y} \frac{\partial \psi_j^{(1)}}{\partial x} \Bigg) \\
& + \frac{\partial \varphi_i}{\partial y} \Bigg(A_{22} \frac{\partial w}{\partial y} \frac{\partial \psi_j^{(1)}}{\partial y} + A_{66} \frac{\partial w}{\partial x} \frac{\partial \psi_j^{(1)}}{\partial x} + A_{26} \frac{\partial w}{\partial x} \frac{\partial \psi_j^{(1)}}{\partial y} + A_{26} \frac{\partial w}{\partial x} \frac{\partial \psi_j^{(1)}}{\partial x} \Bigg) \\
& + \frac{\partial^2 \varphi_i}{\partial x^2} \Bigg(-c_1 E_{12} \frac{\partial \psi_j^{(1)}}{\partial y} - c_1 E_{16} \frac{\partial \psi_j^{(1)}}{\partial x} \Bigg) + \frac{\partial^2 \varphi_i}{\partial y^2} \Bigg(-c_1 E_{22} \frac{\partial \psi_j^{(1)}}{\partial y} - c_1 E_{26} \frac{\partial \psi_j^{(1)}}{\partial x} \Bigg) \\
& + \frac{\partial^2 \varphi_i}{\partial x \partial y} \Bigg(-2 c_1 E_{26} \frac{\partial \psi_j^{(1)}}{\partial y} - 2 c_1 E_{66} \frac{\partial \psi_j^{(1)}}{\partial x} \Bigg) \Bigg] dxdy
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
K_{ij}^{33} = \int_{\Omega^e} \Bigg[& \frac{\partial \varphi_i}{\partial x} \Bigg(\frac{1}{2} A_{11} (\frac{\partial w}{\partial x})^2 \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{12} (\frac{\partial w}{\partial y})^2 \frac{\partial \varphi_j}{\partial y} + \frac{1}{2} A_{16} \frac{\partial w}{\partial x} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y}) \\
& + \frac{1}{2} A_{16} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{26} (\frac{\partial w}{\partial y})^2 \frac{\partial \varphi_j}{\partial y} + \frac{1}{2} A_{66} \frac{\partial w}{\partial y} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) + A_{45} \frac{\partial \varphi_j}{\partial y} \\
& + A_{55} \frac{\partial \varphi_j}{\partial x} - 2 c_2 D_{45} \frac{\partial \varphi_j}{\partial y} - 2 c_2 D_{55} \frac{\partial \varphi_j}{\partial x} + c_2^2 F_{45} \frac{\partial \varphi_j}{\partial y} + c_2^2 F_{55} \frac{\partial \varphi_j}{\partial x} - c_1 E_{11} \frac{\partial^2 \varphi_j}{\partial x^2} \frac{\partial w}{\partial x} \\
& - c_1 E_{12} \frac{\partial^2 \varphi_j}{\partial y^2} \frac{\partial w}{\partial x} - 2 c_1 E_{16} \frac{\partial^2 \varphi_j}{\partial x \partial y} \frac{\partial w}{\partial x} - c_1 E_{16} \frac{\partial^2 \varphi_j}{\partial x^2} \frac{\partial w}{\partial y} - c_1 E_{26} \frac{\partial^2 \varphi_j}{\partial y^2} \frac{\partial w}{\partial y} \\
& - 2 c_1 E_{66} \frac{\partial^2 \varphi_j}{\partial x \partial y} \frac{\partial w}{\partial y} \Bigg) + \frac{\partial \varphi_i}{\partial y} \Bigg(\frac{1}{2} A_{16} (\frac{\partial w}{\partial x})^2 \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{26} \frac{\partial w}{\partial y} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} \\
& + \frac{1}{2} A_{66} \frac{\partial w}{\partial x} (\frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) + \frac{1}{2} A_{12} (\frac{\partial w}{\partial x})^2 \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{22} (\frac{\partial w}{\partial y})^2 \frac{\partial \varphi_j}{\partial y} \\
& + \frac{1}{2} A_{26} \frac{\partial w}{\partial y} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) + A_{44} \frac{\partial \varphi_j}{\partial y} + A_{45} \frac{\partial \varphi_j}{\partial x} - 2 c_2 D_{44} \frac{\partial \varphi_j}{\partial y} - 2 c_2 D_{45} \frac{\partial \varphi_j}{\partial x} \\
& + c_2^2 F_{44} \frac{\partial \varphi_j}{\partial y} + c_2^2 F_{45} \frac{\partial \varphi_j}{\partial x} - c_1 E_{12} \frac{\partial^2 \varphi_j}{\partial y^2} \frac{\partial w}{\partial x} - c_1 E_{12} \frac{\partial^2 \varphi_j}{\partial x^2} \frac{\partial w}{\partial y} - 2 c_1 E_{26} \frac{\partial^2 \varphi_j}{\partial x \partial y} \frac{\partial w}{\partial y} \\
& - c_1 E_{16} \frac{\partial^2 \varphi_j}{\partial x^2} \frac{\partial w}{\partial x} - c_1 E_{26} \frac{\partial^2 \varphi_j}{\partial y^2} \frac{\partial w}{\partial x} - 2 c_1 E_{66} \frac{\partial^2 \varphi_j}{\partial x \partial y} \frac{\partial w}{\partial x} \Bigg) + \frac{\partial^2 \varphi_i}{\partial x^2} \Bigg(-\frac{1}{2} c_1 E_{11} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} \\
& - \frac{1}{2} c_1 E_{12} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} - \frac{1}{2} c_1 E_{16} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) + c_1^2 H_{11} \frac{\partial^2 \varphi_j}{\partial x^2} + c_1^2 H_{12} \frac{\partial^2 \varphi_j}{\partial y^2} \\
& + 2 c_1^2 H_{16} \frac{\partial^2 \varphi_j}{\partial x \partial y} \Bigg) + \frac{\partial^2 \varphi_i}{\partial y^2} \Bigg(-\frac{1}{2} c_1 E_{12} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} - \frac{1}{2} c_1 E_{22} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} \\
& - \frac{1}{2} c_1 E_{26} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) + c_1^2 H_{12} \frac{\partial^2 \varphi_j}{\partial x^2} + c_1^2 H_{22} \frac{\partial^2 \varphi_j}{\partial y^2} + 2 c_1^2 H_{26} \frac{\partial^2 \varphi_j}{\partial x \partial y} \Bigg)
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
K_{ij}^{34} = \int_{\Omega^e} \Bigg[& \frac{\partial \varphi_i}{\partial x} \Bigg(A_{55}\psi_j^{(2)} - 2c_2 D_{55}\psi_j^{(2)} + c_2^2 F_{55}\psi_j^{(2)} + (B_{11} - c_1 E_{11}) \frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} \\
& + (B_{16} - c_1 E_{16}) \frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + (B_{16} - c_1 E_{16}) \frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x} + (B_{66} - c_1 E_{66}) \frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} \Bigg) \\
& + \frac{\partial \varphi_i}{\partial y} \Bigg(A_{45}\psi_j^{(2)} - 2c_2 D_{45}\psi_j^{(2)} + c_2^2 F_{45}\psi_j^{(2)} + (B_{12} - c_1 E_{12}) \frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x} \\
& + (B_{26} - c_1 E_{26}) \frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} + (B_{16} - c_1 E_{16}) \frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + (B_{66} - c_1 E_{66}) \frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} \Bigg) \\
& + \frac{\partial^2 \varphi_i}{\partial x^2} \Bigg(-c_1 F_{11} \frac{\partial \psi_j^{(2)}}{\partial x} - c_1 F_{16} \frac{\partial \psi_j^{(2)}}{\partial y} + c_1^2 H_{11} \frac{\partial \psi_j^{(2)}}{\partial x} + c_1^2 H_{16} \frac{\partial \psi_j^{(2)}}{\partial y} \Bigg) \\
& + \frac{\partial^2 \varphi_i}{\partial y^2} \Bigg(-c_1 F_{12} \frac{\partial \psi_j^{(2)}}{\partial x} - c_1 F_{26} \frac{\partial \psi_j^{(2)}}{\partial y} + c_1^2 H_{12} \frac{\partial \psi_j^{(2)}}{\partial x} + c_1^2 H_{26} \frac{\partial \psi_j^{(2)}}{\partial y} \Bigg) \\
& + \frac{\partial^2 \varphi_i}{\partial x \partial y} \Bigg(-2c_1 F_{16} \frac{\partial \psi_j^{(2)}}{\partial x} - 2c_1 F_{66} \frac{\partial \psi_j^{(2)}}{\partial y} + 2c_1^2 H_{16} \frac{\partial \psi_j^{(2)}}{\partial x} + 2c_1^2 H_{66} \frac{\partial \psi_j^{(2)}}{\partial y} \Bigg) \Bigg] dxdy
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
K_{ij}^{35} = \int_{\Omega^e} \Bigg[& \frac{\partial \varphi_i}{\partial x} \Bigg(A_{45}\psi_j^{(2)} - 2c_2 D_{45}\psi_j^{(2)} + c_2^2 F_{45}\psi_j^{(2)} + (B_{12} - c_1 E_{12}) \frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} \\
& + (B_{16} - c_1 E_{16}) \frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} + (B_{26} - c_1 E_{26}) \frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} + (B_{66} - c_1 E_{66}) \frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x} \Bigg) \\
& + \frac{\partial \varphi_i}{\partial y} \Bigg(A_{44}\psi_j^{(2)} - 2c_2 D_{44}\psi_j^{(2)} + c_2^2 F_{44}\psi_j^{(2)} + (B_{22} - c_1 E_{22}) \frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial y} \\
& + (B_{26} - c_1 E_{26}) \frac{\partial w}{\partial y}\frac{\partial \psi_j^{(2)}}{\partial x} + (B_{26} - c_1 E_{26}) \frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial y} + (B_{66} - c_1 E_{66}) \frac{\partial w}{\partial x}\frac{\partial \psi_j^{(2)}}{\partial x} \Bigg) \\
& + \frac{\partial^2 \varphi_i}{\partial x^2} \Bigg(-c_1 F_{12} \frac{\partial \psi_j^{(2)}}{\partial y} - c_1 F_{16} \frac{\partial \psi_j^{(2)}}{\partial x} + c_1^2 H_{12} \frac{\partial \psi_j^{(2)}}{\partial y} + c_1^2 H_{16} \frac{\partial \psi_j^{(2)}}{\partial x} \Bigg) \\
& + \frac{\partial^2 \varphi_i}{\partial y^2} \Bigg(-c_1 F_{22} \frac{\partial \psi_j^{(2)}}{\partial y} - c_1 F_{26} \frac{\partial \psi_j^{(2)}}{\partial x} + c_1^2 H_{22} \frac{\partial \psi_j^{(2)}}{\partial y} + c_1^2 H_{26} \frac{\partial \psi_j^{(2)}}{\partial x} \Bigg) \\
& + \frac{\partial^2 \varphi_i}{\partial x \partial y} \Bigg(-2c_1 F_{26} \frac{\partial \psi_j^{(2)}}{\partial y} - 2c_1 F_{66} \frac{\partial \psi_j^{(2)}}{\partial x} + 2c_1^2 H_{26} \frac{\partial \psi_j^{(2)}}{\partial y} + 2c_1^2 H_{66} \frac{\partial \psi_j^{(2)}}{\partial x} \Bigg) \Bigg] dxdy
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
K_{ij}^{41} = \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(2)}}{\partial x} \Bigg(B_{11} \frac{\partial \psi_j^{(1)}}{\partial x} + B_{16} \frac{\partial \psi_j^{(1)}}{\partial y} - c_1 E_{11} \frac{\partial \psi_j^{(1)}}{\partial x} - c_1 E_{16} \frac{\partial \psi_j^{(1)}}{\partial y} \Bigg) \\
& + \frac{\partial \psi_i^{(2)}}{\partial y} \Bigg(B_{16} \frac{\partial \psi_j^{(1)}}{\partial x} + B_{66} \frac{\partial \psi_j^{(1)}}{\partial y} - c_1 E_{16} \frac{\partial \psi_j^{(1)}}{\partial x} - c_1 E_{66} \frac{\partial \psi_j^{(1)}}{\partial y} \Bigg) \Bigg] dxdy
\end{aligned} \tag{66}
$$

$$K_{ij}^{42} = \int_{\Omega^e} \bigg[\frac{\partial \psi_i^{(2)}}{\partial x} \bigg(B_{12} \frac{\partial \psi_j^{(1)}}{\partial y} + B_{16} \frac{\partial \psi_j^{(1)}}{\partial x} - c_1 E_{12} \frac{\partial \psi_j^{(1)}}{\partial y} - c_1 E_{16} \frac{\partial \psi_j^{(1)}}{\partial x} \bigg) + \frac{\partial \psi_i^{(2)}}{\partial y} \bigg(B_{26} \frac{\partial \psi_j^{(1)}}{\partial y} + B_{66} \frac{\partial \psi_j^{(1)}}{\partial x} - c_1 E_{26} \frac{\partial \psi_j^{(1)}}{\partial y} - c_1 E_{66} \frac{\partial \psi_j^{(1)}}{\partial x} \bigg) \bigg] dxdy \tag{67}$$

$$\begin{aligned} K_{ij}^{43} = \int_{\Omega^e} \bigg[& \frac{\partial \psi_i^{(2)}}{\partial x} \bigg(\frac{1}{2} B_{11} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} B_{12} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + \frac{1}{2} B_{16} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 \frac{1}{2} E_{11} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} \\ & - c_1 \frac{1}{2} E_{12} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} - \frac{1}{2} c_1 E_{16} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 F_{11} \frac{\partial^2 \varphi_j}{\partial x^2} - c_1 F_{12} \frac{\partial^2 \varphi_j}{\partial y^2} \\ & - 2c_1 F_{16} \frac{\partial^2 \varphi_j}{\partial x \partial y} + c_1^2 H_{11} \frac{\partial^2 \varphi_j}{\partial x^2} + c_1^2 H_{12} \frac{\partial^2 \varphi_j}{\partial y^2} + 2c_1^2 H_{16} \frac{\partial^2 \varphi_j}{\partial x \partial y} \bigg) + \frac{\partial \psi_i^{(2)}}{\partial y} \bigg(\frac{1}{2} B_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} \\ & + \frac{1}{2} B_{26} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + \frac{1}{2} B_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 \frac{1}{2} E_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} - c_1 \frac{1}{2} E_{26} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} \\ & - \frac{1}{2} c_1 E_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 F_{16} \frac{\partial^2 \varphi_j}{\partial x^2} - c_1 F_{26} \frac{\partial^2 \varphi_j}{\partial y^2} - 2c_1 F_{66} \frac{\partial^2 \varphi_j}{\partial x \partial y} + c_1^2 H_{16} \frac{\partial^2 \varphi_j}{\partial x^2} \\ & + c_1^2 H_{26} \frac{\partial^2 \varphi_j}{\partial y^2} + 2c_1^2 H_{66} \frac{\partial^2 \varphi_j}{\partial x \partial y} \bigg) + \psi_i^{(2)} \bigg(A_{45} \frac{\partial \varphi_j}{\partial y} + A_{55} \frac{\partial \varphi_j}{\partial x} - 2c_2 D_{45} \frac{\partial \varphi_j}{\partial y} - 2c_2 D_{55} \frac{\partial \varphi_j}{\partial x} \\ & + c_2^2 F_{45} \frac{\partial \varphi_j}{\partial y} + c_2^2 F_{55} \frac{\partial \varphi_j}{\partial x} \bigg) \bigg] dxdy \end{aligned} \tag{68}$$

$$\begin{aligned} K_{ij}^{44} = \int_{\Omega^e} \bigg[& \frac{\partial \psi_i^{(2)}}{\partial x} \bigg(D_{11} \frac{\partial \psi_j^{(2)}}{\partial x} + D_{16} \frac{\partial \psi_j^{(2)}}{\partial y} - 2c_1 F_{11} \frac{\partial \psi_j^{(2)}}{\partial x} - 2c_1 F_{16} \frac{\partial \psi_j^{(2)}}{\partial y} + c_1^2 H_{11} \frac{\partial \psi_j^{(2)}}{\partial x} \\ & + c_1^2 H_{16} \frac{\partial \psi_j^{(2)}}{\partial y} \bigg) + \frac{\partial \psi_i^{(2)}}{\partial y} \bigg(D_{16} \frac{\partial \psi_j^{(2)}}{\partial x} + D_{66} \frac{\partial \psi_j^{(2)}}{\partial y} - 2c_1 F_{16} \frac{\partial \psi_j^{(2)}}{\partial x} - 2c_1 F_{66} \frac{\partial \psi_j^{(2)}}{\partial y} \\ & + c_1^2 H_{16} \frac{\partial \psi_j^{(2)}}{\partial x} + c_1^2 H_{66} \frac{\partial \psi_j^{(2)}}{\partial y} \bigg) + \psi_i^{(2)} \bigg(A_{55} \psi_j^{(2)} - 2c_2 D_{55} \psi_j^{(2)} + c_2^2 F_{55} \psi_j^{(2)} \bigg) \bigg] dxdy \end{aligned} \tag{69}$$

$$\begin{aligned} K_{ij}^{45} = \int_{\Omega^e} \bigg[& \frac{\partial \psi_i^{(2)}}{\partial x} \bigg(D_{12} \frac{\partial \psi_j^{(2)}}{\partial y} + D_{16} \frac{\partial \psi_j^{(2)}}{\partial x} - 2c_1 F_{12} \frac{\partial \psi_j^{(2)}}{\partial y} - 2c_1 F_{16} \frac{\partial \psi_j^{(2)}}{\partial x} + c_1^2 H_{12} \frac{\partial \psi_j^{(2)}}{\partial y} \\ & + c_1^2 H_{16} \frac{\partial \psi_j^{(2)}}{\partial x} \bigg) + \frac{\partial \psi_i^{(2)}}{\partial y} \bigg(D_{26} \frac{\partial \psi_j^{(2)}}{\partial y} + D_{66} \frac{\partial \psi_j^{(2)}}{\partial x} - 2c_1 F_{26} \frac{\partial \psi_j^{(2)}}{\partial y} - 2c_1 F_{66} \frac{\partial \psi_j^{(2)}}{\partial x} \\ & + c_1^2 H_{26} \frac{\partial \psi_j^{(2)}}{\partial y} + c_1^2 H_{66} \frac{\partial \psi_j^{(2)}}{\partial x} \bigg) + \psi_i^{(2)} (A_{45} \psi_j^{(2)} - 2c_2 D_{45} \psi_j^{(2)} + c_2^2 F_{45} \psi_j^{(2)}) \bigg] dxdy \end{aligned} \tag{70}$$

$$K_{ij}^{51} = \int_{\Omega^e} \bigg[\frac{\partial \psi_i^{(2)}}{\partial y} \bigg(B_{12} \frac{\partial \psi_j^{(1)}}{\partial x} + B_{26} \frac{\partial \psi_j^{(1)}}{\partial y} - c_1 E_{12} \frac{\partial \psi_j^{(1)}}{\partial x} - c_1 E_{26} \frac{\partial \psi_j^{(1)}}{\partial y} \bigg) + \frac{\partial \psi_i^{(2)}}{\partial x} \bigg(B_{16} \frac{\partial \psi_j^{(1)}}{\partial x} + B_{66} \frac{\partial \psi_j^{(1)}}{\partial y} - c_1 E_{16} \frac{\partial \psi_j^{(1)}}{\partial x} - c_1 E_{66} \frac{\partial \psi_j^{(1)}}{\partial y} \bigg) \bigg] dxdy \tag{71}$$

$$K_{ij}^{52} = \int_{\Omega^e} \Bigg[\frac{\partial \psi_i^{(2)}}{\partial y} \Bigg(B_{22} \frac{\partial \psi_j^{(1)}}{\partial y} + B_{26} \frac{\partial \psi_j^{(1)}}{\partial x} - c_1 E_{22} \frac{\partial \psi_j^{(1)}}{\partial y} - c_1 E_{26} \frac{\partial \psi_j^{(1)}}{\partial x} \Bigg) + \frac{\partial \psi_i^{(2)}}{\partial x} \Bigg(B_{26} \frac{\partial \psi_j^{(1)}}{\partial y} + B_{66} \frac{\partial \psi_j^{(1)}}{\partial x} - c_1 E_{26} \frac{\partial \psi_j^{(1)}}{\partial y} - c_1 E_{66} \frac{\partial \psi_j^{(1)}}{\partial x} \Bigg) \Bigg] dxdy \tag{72}$$

$$\begin{aligned} K_{ij}^{53} = \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(2)}}{\partial x} \Bigg(\frac{1}{2} B_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} B_{26} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + B_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 \frac{1}{2} E_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} \\ & - c_1 \frac{1}{2} E_{26} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} - c_1 E_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 F_{16} \frac{\partial^2 \varphi_j}{\partial x^2} - c_1 F_{26} \frac{\partial^2 \varphi_j}{\partial y^2} \\ & - 2 c_1 F_{66} \frac{\partial^2 \varphi_j}{\partial x \partial y} + c_1^2 H_{16} \frac{\partial^2 \varphi_j}{\partial x^2} + c_1^2 H_{26} \frac{\partial^2 \varphi_j}{\partial y^2} + 2 c_1^2 H_{66} \frac{\partial^2 \varphi_j}{\partial x \partial y} \Bigg) \\ & + \frac{\partial \psi_i^{(2)}}{\partial y} \Bigg(\frac{1}{2} B_{12} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} B_{22} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + B_{26} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 \frac{1}{2} E_{12} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} \\ & - c_1 \frac{1}{2} E_{22} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} - c_1 E_{26} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 F_{12} \frac{\partial^2 \varphi_j}{\partial x^2} - c_1 F_{22} \frac{\partial^2 \varphi_j}{\partial y^2} \\ & - 2 c_1 F_{26} \frac{\partial^2 \varphi_j}{\partial x \partial y} + c_1^2 H_{12} \frac{\partial^2 \varphi_j}{\partial x^2} + c_1^2 H_{22} \frac{\partial^2 \varphi_j}{\partial y^2} + 2 c_1^2 H_{26} \frac{\partial^2 \varphi_j}{\partial x \partial y} \Bigg) \\ & + \psi_i^{(2)} (A_{44} \frac{\partial \varphi_j}{\partial y} + A_{45} \frac{\partial \varphi_j}{\partial x} - 2 c_2 D_{44} \frac{\partial \varphi_j}{\partial y} - 2 c_2 D_{45} \frac{\partial \varphi_j}{\partial x} \\ & + c_2^2 F_{44} \frac{\partial \varphi_j}{\partial y} + c_2^2 F_{45} \frac{\partial \varphi_j}{\partial x}) \Bigg] dxdy \end{aligned} \tag{73}$$

$$\begin{aligned} K_{ij}^{54} = \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(2)}}{\partial y} \Bigg(D_{12} \frac{\partial \psi_j^{(2)}}{\partial x} + D_{26} \frac{\partial \psi_j^{(2)}}{\partial y} - 2 c_1 F_{12} \frac{\partial \psi_j^{(2)}}{\partial x} - 2 c_1 F_{26} \frac{\partial \psi_j^{(2)}}{\partial y} + c_1^2 H_{12} \frac{\partial \psi_j^{(2)}}{\partial x} \\ & + c_1^2 H_{26} \frac{\partial \psi_j^{(2)}}{\partial y} \Bigg) + \frac{\partial \psi_i^{(2)}}{\partial x} \Bigg(D_{16} \frac{\partial \psi_j^{(2)}}{\partial x} + D_{66} \frac{\partial \psi_j^{(2)}}{\partial y} - 2 c_1 F_{16} \frac{\partial \psi_j^{(2)}}{\partial x} - 2 c_1 F_{66} \frac{\partial \psi_j^{(2)}}{\partial y} \\ & + c_1^2 H_{16} \frac{\partial \psi_j^{(2)}}{\partial x} + c_1^2 H_{66} \frac{\partial \psi_j^{(2)}}{\partial y} \Bigg) + \psi_i^{(2)} \Bigg(A_{45} \psi_j^{(2)} - 2 c_2 D_{45} \psi_j^{(2)} + c_2^2 F_{45} \psi_j^{(2)} \Bigg) \Bigg] dxdy \end{aligned} \tag{74}$$

$$\begin{aligned} K_{ij}^{55} = \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(2)}}{\partial y} \Bigg(D_{22} \frac{\partial \psi_j^{(2)}}{\partial y} + D_{26} \frac{\partial \psi_j^{(2)}}{\partial x} - 2 c_1 F_{22} \frac{\partial \psi_j^{(2)}}{\partial y} - 2 c_1 F_{26} \frac{\partial \psi_j^{(2)}}{\partial x} + c_1^2 H_{22} \frac{\partial \psi_j^{(2)}}{\partial y} \\ & + c_1^2 H_{26} \frac{\partial \psi_j^{(2)}}{\partial x} \Bigg) + \frac{\partial \psi_i^{(2)}}{\partial x} \Bigg(D_{26} \frac{\partial \psi_j^{(2)}}{\partial y} + D_{66} \frac{\partial \psi_j^{(2)}}{\partial x} - 2 c_1 F_{26} \frac{\partial \psi_j^{(2)}}{\partial y} - 2 c_1 F_{66} \frac{\partial \psi_j^{(2)}}{\partial x} \\ & + c_1^2 H_{26} \frac{\partial \psi_j^{(2)}}{\partial y} + c_1^2 H_{66} \frac{\partial \psi_j^{(2)}}{\partial x} \Bigg) + \psi_i^{(2)} (A_{44} \psi_j^{(2)} - 2 c_2 D_{44} \psi_j^{(2)} + c_2^2 F_{44} \psi_j^{(2)}) \Bigg] dxdy \end{aligned} \tag{75}$$

The elements of the force vector are given by

$$F_i^1 = \oint_{\Gamma^e} (N_{xx}\hat{n}_x + N_{xy}\hat{n}_y)ds \tag{76}$$

$$F_i^2 = \oint_{\Gamma^e} (N_{xy}\hat{n}_x + N_{yy}\hat{n}_y)ds \tag{77}$$

$$\begin{aligned} F_i^3 = \int_{\Omega^e} (1-\mu\nabla^2)q\varphi_i dxdy + \oint_{\Gamma^e} \Big\{ (\bar{Q}_x\hat{n}_x + \bar{Q}_y\hat{n}_y) + (N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y})\hat{n}_x + (N_{xy}\frac{\partial w_0}{\partial x} + \\ N_{yy}\frac{\partial w_0}{\partial y})\hat{n}_y + c_1\Big[(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y})\hat{n}_x + (\frac{\partial P_{yy}}{\partial y} + \frac{\partial P_{xy}}{\partial x})\hat{n}_y \Big] - c_1\Big[(P_{xx} + P_{xy})\hat{n}_x \\ + (P_{yy} + P_{xy})\hat{n}_y \Big] \Big\} ds \end{aligned} \tag{78}$$

$$F_i^4 = \oint_{\Gamma^e} (M_{xx}\hat{n}_x + M_{xy}\hat{n}_y)\psi^{(2)}ds \tag{79}$$

$$F_i^5 = \oint_{\Gamma^e} (M_{xy}\hat{n}_x + M_{yy}\hat{n}_y)\psi^{(2)}ds \tag{80}$$

Solution of nonlinear equations

Solution of Eq. (50) by the Newton iteration method results in the following linearized equations for the incremental solution at the $(r+1)$st iteration:

$$\delta\boldsymbol{\Delta} = -(\hat{\mathbf{T}}(\boldsymbol{\Delta}_{s+1}^r))^{-1}\mathbf{R}_{s+1}^r \tag{81}$$

$$\hat{\mathbf{T}}(\boldsymbol{\Delta}_{s+1}^r) \left[\frac{\partial \mathbf{R}}{\partial \boldsymbol{\Delta}}\right]_{s+1}^r, \quad \mathbf{R}_{s+1}^r = \hat{\mathbf{K}}(\boldsymbol{\Delta}_{s+1}^r)\boldsymbol{\Delta}_{s+1}^r - \hat{\mathbf{F}} \tag{82}$$

The total solution is obtained from

$$\boldsymbol{\Delta}_{s+1}^{r+1} = \boldsymbol{\Delta}_{s+1}^r + \delta\boldsymbol{\Delta}$$

The tangent stiffness coefficients are computed from (see [41])

$$T_{ij}^{\alpha\beta} \equiv \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta} = K_{ij}^{\alpha\beta} + \sum_{k=1}^{n_\gamma} \frac{\partial K_{ik}^{\alpha\gamma}}{\partial \Delta_j^\beta}\Delta_k^\gamma - \frac{\partial F_i^\alpha}{\partial \Delta_j^\beta} \tag{83}$$

Using the above equation the tangent stiffness coeffcients are derived as follows,

$$T_{ij}^{11} = K_{ij}^{11}, \quad T_{ij}^{12} = K_{ij}^{12} \tag{84}$$

$$T_{ij}^{13} = K_{ij}^{13} + \int_{\Omega^e} \left[\frac{\partial \psi_i^{(1)}}{\partial x} \left(\frac{1}{2} A_{11} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{12} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + \frac{1}{2} A_{16} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) \right) + \frac{\partial \psi_i^{(1)}}{\partial y} \left(\frac{1}{2} A_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{26} \frac{\partial w}{\partial y} \frac{\partial \psi_j}{\partial y} + \frac{1}{2} A_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) \right) \right] dxdy \tag{85}$$

$$T_{ij}^{14} = K_{ij}^{14}, \quad T_{ij}^{15} = K_{ij}^{15} \tag{86}$$

$$T_{ij}^{21} = K_{ij}^{21}, \quad T_{ij}^{22} = K_{ij}^{22} \tag{87}$$

$$T_{ij}^{23} = K_{ij}^{23} + \int_{\Omega^e} \left[\frac{\partial \psi_i^{(1)}}{\partial y} \left(\frac{1}{2} A_{12} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{22} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + \frac{1}{2} A_{26} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y}) \right) + \frac{\partial \psi_i^{(1)}}{\partial x} \left(\frac{1}{2} A_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} A_{26} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + \frac{1}{2} A_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y}) \right) \right] dxdy \tag{88}$$

$$T_{ij}^{24} = K_{ij}^{24}, \quad T_{ij}^{25} = K_{ij}^{25} \tag{89}$$

$$T_{ij}^{31} = K_{ij}^{31}, T_{ij}^{32} = K_{ij}^{32} \tag{90}$$

$$\begin{aligned} T_{ij}^{33} = K_{ij}^{33} + \int_{\Omega^e} \Bigg\{ & \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial y} (A_{11} \frac{\partial u}{\partial x} + A_{16} \frac{\partial u}{\partial y}) + \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} (A_{12} \frac{\partial u}{\partial x} + A_{26} \frac{\partial u}{\partial y}) + \left(\frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial y} + \right. \\ & \left. \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial x} \right) \left(A_{11} \frac{\partial u}{\partial x} + A_{16} \frac{\partial u}{\partial y} \right) + \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} (A_{16} \frac{\partial v}{\partial x} + A_{12} \frac{\partial v}{\partial y}) + \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} (A_{26} \frac{\partial v}{\partial x} \\ & + A_{22} \frac{\partial v}{\partial y}) + \left(\frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial x} \right) \left(A_{66} \frac{\partial v}{\partial x} + A_{26} \frac{\partial v}{\partial y} \right) + \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} \left(A_{11} (\frac{\partial w}{\partial x})^2 \right. \\ & \left. + A_{66} (\frac{\partial w}{\partial y})^2 + 2A_{16} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} \left(A_{66} (\frac{\partial w}{\partial x})^2 + A_{22} (\frac{\partial w}{\partial y})^2 + 2A_{26} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \\ & + \left(\frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial x} \right) \left(A_{16} (\frac{\partial w}{\partial x})^2 + A_{26} (\frac{\partial w}{\partial y})^2 + (A_{16} + A_{66}) \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \\ & - c_1 \Bigg[\frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial y} \left(E_{11} \frac{\partial^2 w}{\partial x^2} + E_{12} \frac{\partial^2 w}{\partial y^2} + 2E_{16} \frac{\partial^2 w}{\partial x \partial y} \right) + \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial y} \left(E_{12} \frac{\partial^2 w}{\partial x^2} + E_{22} \frac{\partial^2 w}{\partial y^2} \right. \\ & \left. + 2E_{23} \frac{\partial^2 w}{\partial x \partial y} \right) + \left(\frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial \varphi_i}{\partial y} \frac{\partial \varphi_j}{\partial x} \right) \left(E_{16} \frac{\partial^2 w}{\partial x^2} + E_{26} \frac{\partial^2 w}{\partial y^2} + 2E_{66} \frac{\partial^2 w}{\partial x \partial y} \right) \Bigg] \\ & - \frac{c_1}{2} \Bigg[\frac{\partial w}{\partial x} \left(E_{11} \frac{\partial^2 \varphi_i}{\partial x^2} \frac{\partial \varphi_j}{\partial x} + E_{16} \frac{\partial^2 \varphi_i}{\partial x^2} \frac{\partial \varphi_j}{\partial y} + E_{12} \frac{\partial^2 \varphi_i}{\partial y^2} \frac{\partial \varphi_j}{\partial x} + E_{26} \frac{\partial^2 \varphi_i}{\partial y^2} \frac{\partial \varphi_j}{\partial y} + 2E_{16} \frac{\partial^2 \varphi_i}{\partial x \partial y} \frac{\partial \varphi_j}{\partial x} \right. \\ & \left. + 2E_{66} \frac{\partial^2 \varphi_j}{\partial x \partial y} \frac{\partial \varphi_j}{\partial y} \right) + \frac{\partial w}{\partial y} \left(E_{12} \frac{\partial^2 \varphi_i}{\partial x^2} \frac{\partial \varphi_j}{\partial y} + E_{16} \frac{\partial^2 \varphi_i}{\partial x^2} \frac{\partial \varphi_j}{\partial x} + E_{22} \frac{\partial^2 \varphi_i}{\partial y^2} \frac{\partial \varphi_j}{\partial y} + E_{26} \frac{\partial^2 \varphi_i}{\partial y^2} \frac{\partial \varphi_j}{\partial x} \right. \\ & \left. + 2E_{26} \frac{\partial^2 \varphi_i}{\partial x \partial y} \frac{\partial \varphi_j}{\partial y} + 2E_{66} \frac{\partial^2 \varphi_i}{\partial x \partial y} \frac{\partial \varphi_j}{\partial x} \right) \Bigg] \Bigg\} dxdy \end{aligned} \tag{91}$$

$$T_{ij}^{34} = K_{ij}^{34}, \quad T_{ij}^{35} = K_{ij}^{35} \tag{92}$$

$$T_{ij}^{41} = K_{ij}^{41}, \quad T_{ij}^{42} = K_{ij}^{42} \tag{93}$$

$$\begin{aligned}
T_{ij}^{43} = K_{ij}^{43} + \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(2)}}{\partial x} \Bigg(\frac{1}{2} B_{11} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} B_{12} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + B_{16} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) \\
& - c_1 \frac{1}{2} E_{11} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} - c_1 \frac{1}{2} E_{12} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} - c_1 E_{16} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) \Bigg) \\
& + \frac{\partial \psi_i^{(2)}}{\partial y} \Bigg(\frac{1}{2} B_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} B_{26} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + B_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 \frac{1}{2} E_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} \\
& - c_1 \frac{1}{2} E_{26} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} - c_1 E_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) \Bigg) \Bigg] dxdy
\end{aligned} \tag{94}$$

$$T_{ij}^{44} = K_{ij}^{44}, \quad T_{ij}^{45} = K_{ij}^{45} \tag{95}$$

$$T_{ij}^{51} = K_{ij}^{51}, \quad T_{ij}^{52} = K_{ij}^{52} \tag{96}$$

$$\begin{aligned}
\mathrm{T}_{ij}^{53} = K_{ij}^{53} + \int_{\Omega^e} \Bigg[& \frac{\partial \psi_i^{(2)}}{\partial x} \Bigg(\frac{1}{2} B_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} B_{26} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + B_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) \\
& - c_1 \frac{1}{2} E_{16} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} - c_1 \frac{1}{2} E_{26} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} - c_1 E_{66} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) \Bigg) \\
& + \frac{\partial \psi_i^{(2)}}{\partial y} \Bigg(\frac{1}{2} B_{12} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} + \frac{1}{2} B_{22} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} + B_{26} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) - c_1 \frac{1}{2} E_{12} \frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial x} \\
& - c_1 \frac{1}{2} E_{22} \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial y} - c_1 E_{26} (\frac{\partial w}{\partial x} \frac{\partial \varphi_j}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j}{\partial x}) \Bigg) \Bigg] dxdy
\end{aligned}$$

$$T_{ij}^{54} = K_{ij}^{54}, \quad T_{ij}^{55} = K_{ij}^{55} \tag{97}$$

Numerical Results

Numerical results are presented to illustrate the effects of nonlocality and nonlinearity on the bending behaviour of laminated composite plates for various boundary conditions and lamination schemes. Three different boundary conditions, namely, SS-1, SS-2, SS-3 are considered for analysis (the SS-1, SS-2 and SS-3 boundary conditions are shown in Figures 3, 4, and 5, respectively). Nonlinear deflections for various values of the nonlocal parameter are obtained. Effect of the lamination scheme on the bending behavior is also studied. The four-noded rectangular element that has 8 degrees freedom at each node, as shown in Figure 2, is used in the analysis.

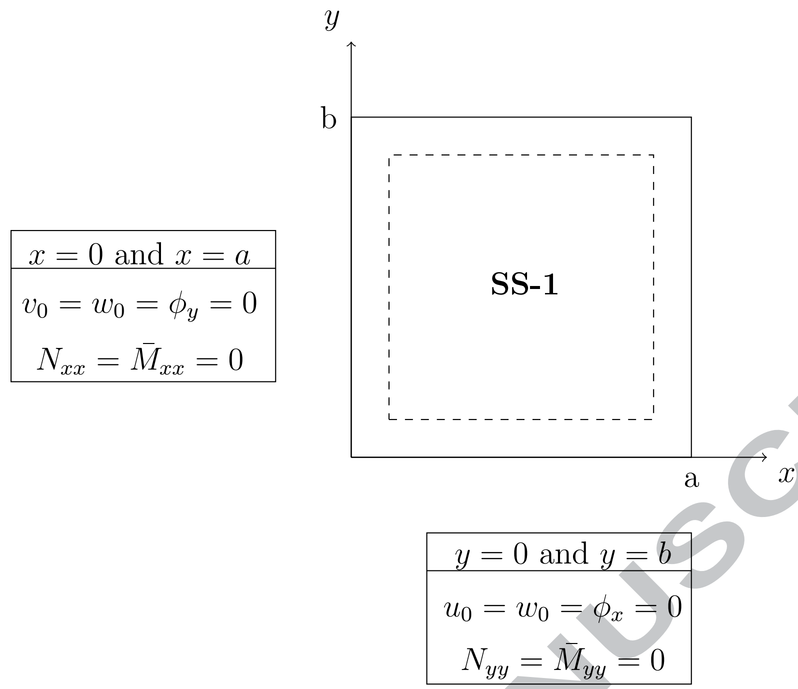

Figure 3: SS-1 Boundary conditions

In all problems considered here, a 4 × 4 mesh with selective integration and a/h ratio of 10 has been considered. A error tolerance value of 10^{-3} is used. The dimensionless center deflection is defined as follows:

$$\bar{w} = \frac{w(\frac{a}{2}, \frac{b}{2}, 0)E_2h^3}{q_0a^4} \tag{98}$$

where a, b, h are the length, breadth, and thickness of the plate, respectively, and q_0 is the intensity of the distributed transverse load.

Example 1

A four-layer, square, symmetric cross-ply (0°/90°/90°/0°) laminated plate is considered in the analysis. The SS-1 and SS-3 boundary conditions are considered. A transverse sinusoidal load of unit intensity is considered to be acting on the plate. The thickness of the each layer is considered equal and the total laminate thickness (h) in all cases is the same. The following material properties are used in the numerical calculations:

$$E_1/E_2 = 25,\ G_{12}/E_2 = 0.5,\ G_{23}/E_2 = 0.2,\ G_{12} = G_{13},\ \nu_{12} = 0.25,\ \nu_{12} = \nu_{13}$$

Table 1 shows the values of dimensionless center deflection with two mesh discretizations, namely 2×2 and 4×4; side-to-thickness ratio a/h of 10 and 20 are considered. Value of the nonlocal parameter μ is increased from 0 to 5. The deflections (with $\mu = 0$) obtained from

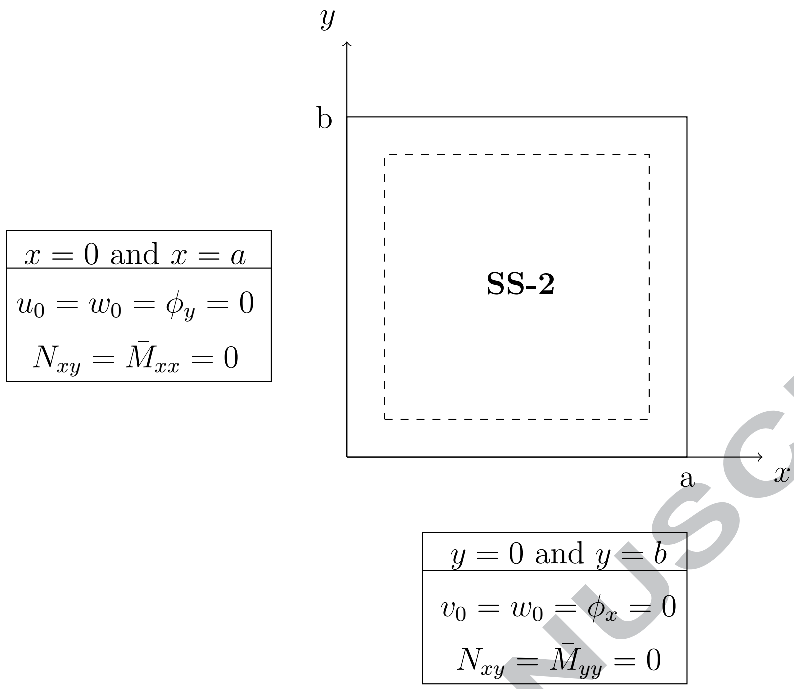

Figure 4: SS-2 Boundary conditions

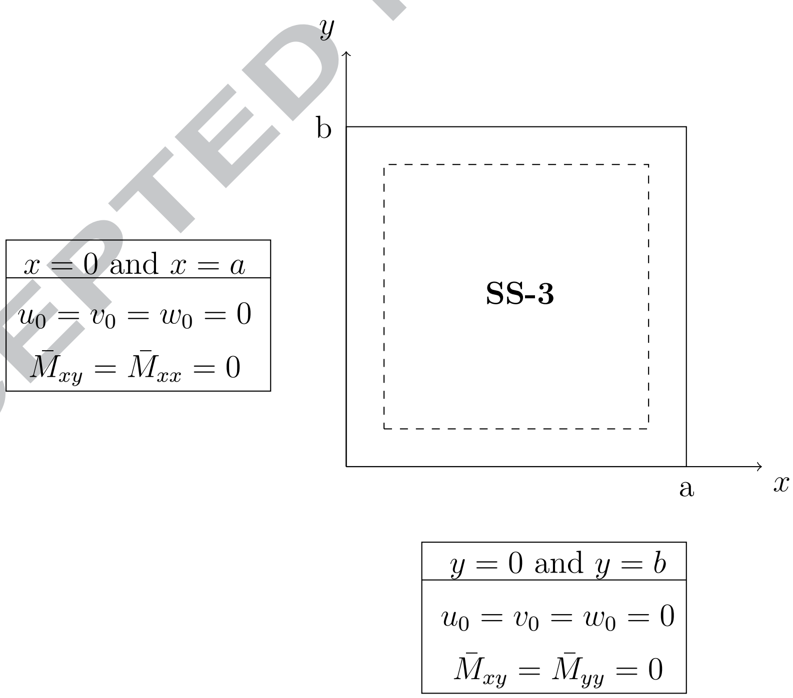

Figure 5: SS-3 Boundary conditions

Table 1: Effect of the reduced integration and nonlocality on the dimensionless center deflection of simply supported (SS-1) symmetric cross-ply laminated plate (0°/90°/90°/0°) subjected to sinusoidal load.

a/h	Source	$\bar{w}$ (Phan et al. [31])	$\bar{w}$ ($\mu = 0$)	$\bar{w}$ ($\mu = 1$)	$\bar{w}$ ($\mu = 3$)	$\bar{w}$ ($\mu = 5$)
10	2E-F	0.7239	0.7205	0.8628	1.1470	1.4310
	2E-FR	0.7294	0.7281	0.8718	1.1590	1.4460
	2E-R	0.7316	0.7341	0.8791	1.169	1.4588
	4E-F	0.7169	0.7142	0.8552	1.1372	1.4190
	4E-FR	0.7177	0.7150	0.8562	1.1385	1.4208
	4E-R	0.7179	0.7145	0.8566	1.1390	1.4210
20	2E-F	0.5040	0.5000	0.5986	0.7960	0.9934
	2E-FR	0.5119	0.5092	0.6098	0.8108	1.0119
	2E-R	0.5128	0.5103	0.6111	0.8126	1.0140
	4E-F	0.5068	0.5047	0.6044	0.8036	1.0029
	4E-FR	0.5079	0.5060	0.6059	0.8057	1.0055
	4E-R	0.5080	0.5061	0.6060	0.8059	1.0057

*2E - 2×2 mesh, F- Full integration, R- Reduced integration, FR- Selective integration.

Table 2: Nonlinear bending of (0°/90°/90°/0°) laminated plate with SS-1 and SS-3 boundary conditions under sinusoidal load. (A 4×4 mesh with selective integration is considered)

Load	SS-1				SS-3			
	Linear		Nonlinear		Nonlinear		Linear	
	$\mu = 0$	$\mu = 1$	$\mu = 0$	$\mu = 1$	$\mu = 0$	$\mu = 1$	$\mu = 0$	$\mu = 1$
0.005	0.0036	0.0044	0.0034	0.004	0.0028	0.0032	0.0036	0.0044
0.01	0.0073	0.0087	0.0062	0.007	0.0043	0.0048	0.0073	0.0087
0.02	0.0146	0.0174	0.01	0.0112	0.0061	0.0066	0.0146	0.0175
0.03	0.0218	0.0262	0.0128	0.0142	0.0073	0.0079	0.0219	0.0262
0.04	0.0291	0.0349	0.0151	0.0167	0.0082	0.0088	0.0292	0.0349
0.05	0.0364	0.0436	0.0171	0.0188	0.009	0.0097	0.0365	0.0437
0.1	0.0728	0.0872	0.0245	0.0267	0.0117	0.0125	0.073	0.0874
0.25	0.182	0.218	0.0375	0.0406	0.0164	0.0175	0.1824	0.2184
0.5	0.3641	0.4359	0.0505	0.0544	0.0209	0.0223	0.3648	0.4369
0.75	0.5461	0.6539	0.0597	0.0642	0.0241	0.0256	0.5473	06553
1	0.7281	0.8719	0.067	0.0721	0.0266	0.0283	0.7297	0.8737

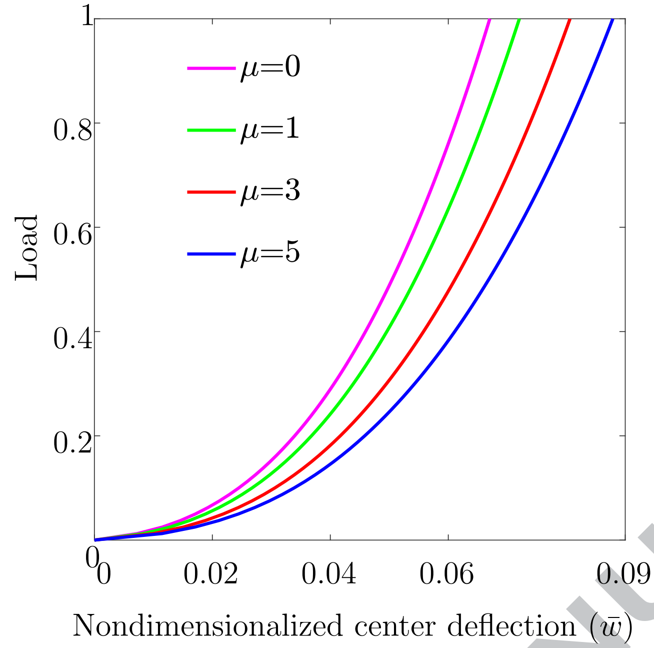

Figure 6: Load versus deflection diagram of ($0°/90°/90°/0°$) plate with SS-1 boundary conditions.

the present study are compared with Phan et al. [31], where the authors have considered 7 degrees of freedom at each node. It is observed that the effect of nonlocality has the significant effect on the deflection value. As the nonlocal parameter μ increases, the value of the deflection increases. The integration rule is also changed from full integration to reduced integration and it is observed that the integration rule does not show significant effect on the deflection.

Table 2 shows the deflection values obtained with the linear and nonlinear analyses, with increasing nonlocal parameters for SS-1 and SS-3 boundary conditions. This study is carried out to check the proximity of the nonlinear deflection values with the linear deflection values in the initial load range. As expected, the difference between the nonlinear and linear deflection increases as the value of the load increases. It is also observed that the SS-3 boundary conditions make the structure more stiffer compared to SS-1 boundary conditions.

Figure 6 shows load versus deflection curves. It is observed that for a given load, the dimensionless center deflection increases as the value of μ increases. It is also observed that, as the value of the load increases, the difference in the deflection with increasing nonlocal parameter increases. Similar observations can be drawn from Figure 7, which shows load-deflection plots of symmetric cross-ply laminated plates ($0°/90°/90°/0°$) for the SS-3 boundary conditions. Comparing Figure 6 with Figure 7, it is observed that the SS-3 boundary conditions make the structure more stiffer than the SS-1 boundary conditions. The increase in the deflection with increase in nonlocal parameter is attributed to reduced structural stiffness (i.e., diffusion type model).

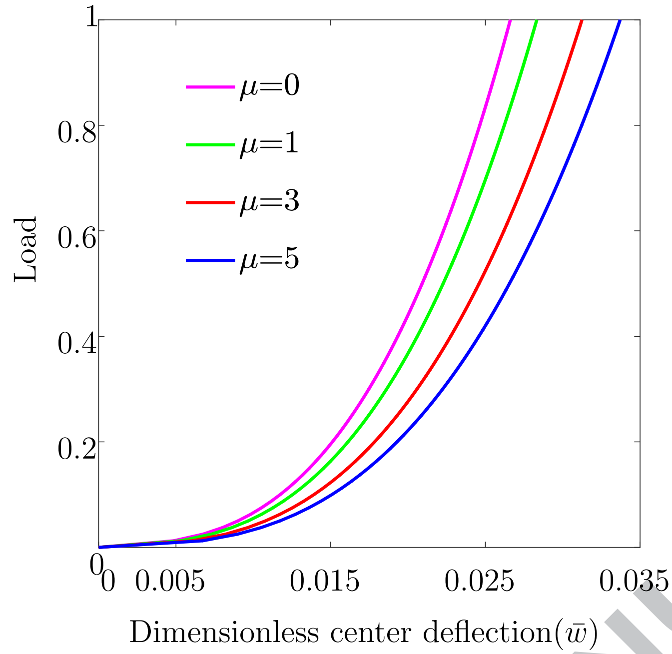

Figure 7: Load versus deflection diagram of (0°/90°/90°/0°) plate with SS-3 boundary conditions.

Example 2

A two-layer square, cross-ply (0°/90°) laminated plate subjected to distributed transverse sinusoidal load of unit intensity is considered in the analysis. The SS-1 and SS-3 boundary conditions are considered. The thickness of the each layer is considered equal and the a/h ratio is considered as 10. The following material properties are used for the analysis:

$$E_1/E_2 = 25,\ G_{12}/E_2 = 0.5,\ G_{23}/E_2 = 0.2,\ G_{12} = G_{13},\ \nu_{12} = 0.25,\ \nu_{12} = \nu_{13}$$

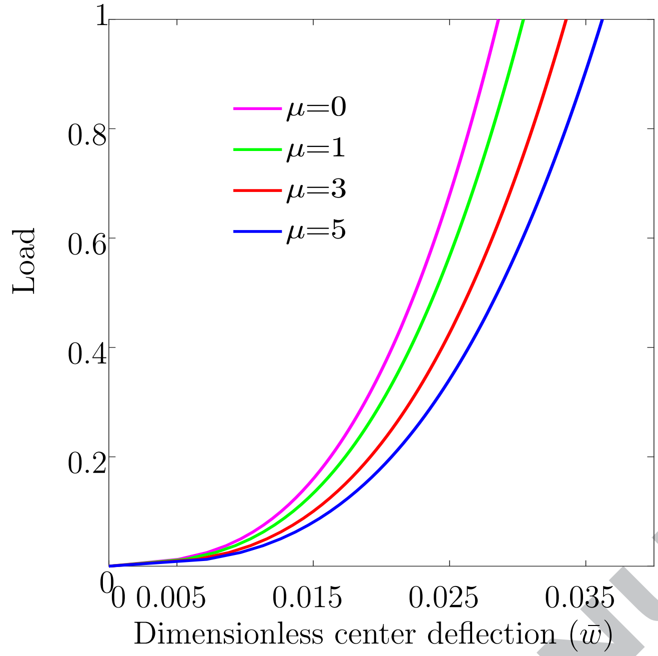

Figure 8: Load versus deflection diagram of (0°/90)plate with SS-3 boundary conditions.

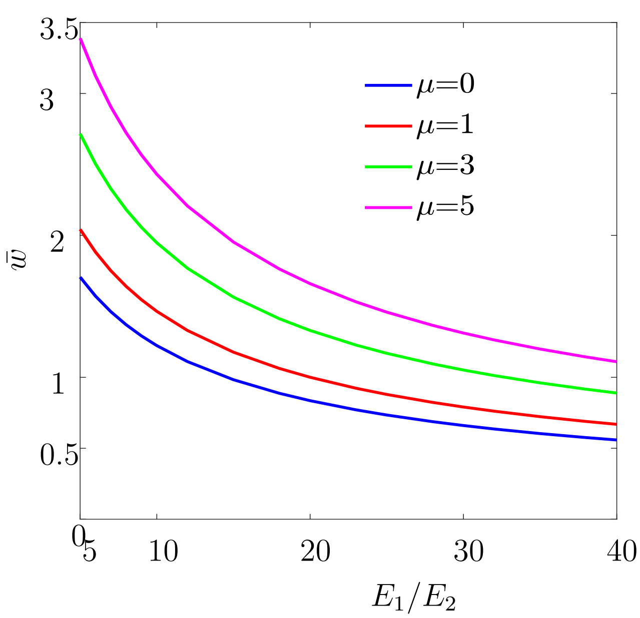

Figure 9: Deflection versus E_1/E_2 diagram of (0°/90) plate with SS-2 boundary conditions.

Figure 8 shows the load versus deflection plot for the (0°/90) plate with the SS-3 boundary

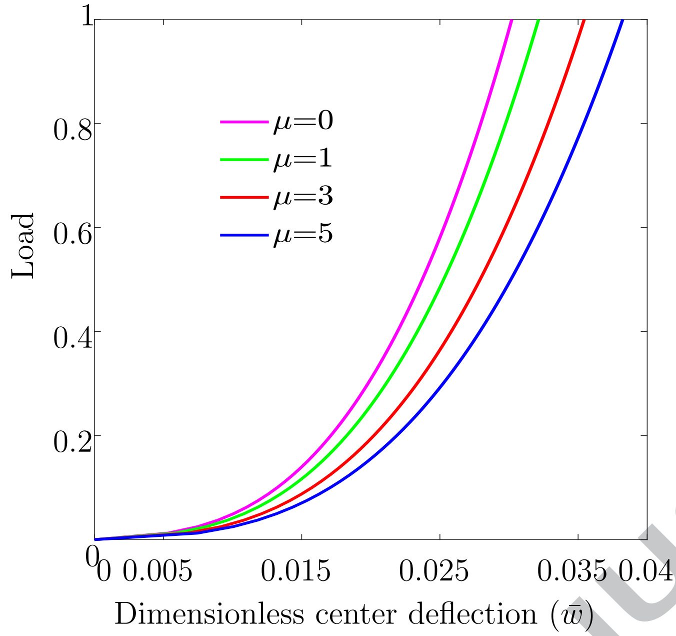

Figure 10: Load versus dimensionless center deflection plot of (-45°/45 °) plate with the SS-2 boundary conditions.

conditions. It is again observed that the effect of nonlocality is to increase the deflection for a given load value. It is also observed from Figures 7 and 8 that the deflection value for a given load is slightly less in the case of (0°/90°) plate compared to (0°/90°/90°/0°) plate with the SS-3 boundary conditions. Figure 9 shows the effect of anisotropy (i.e., E_1/E_2) on the deflections of (0°/90) plate for $a/h = 10$ with with SS-2 boundary conditions. It can be seen that for the given value of E_1/E_2, the value of $\bar{w}$ increases with increase in nonlocal parameter, and as the value of E_1/E_2 increases, the difference between the values of $\bar{w}$ with increasing nonlocal parameter decreases. It is also observed that for a given value of μ, the value of $\bar{w}$ decreases with increase in E_1/E_2 ratio.

Example 3

A two-layer square (-45°/45°) angle-ply laminated plate subjected to distributed transverse sinusoidal load of unit intensity is considered for the analysis. The SS-2 and SS-3 boundary conditions are considered. The thickness of the each layer is considered equal. The following material properties are used:

$$E_1/E_2 = 25,\ G_{12}/E_2 = 0.6,\ G_{23}/E_2 = 0.5,\ G_{12} = G_{13},\ \nu_{12} = 0.25,\ \nu_{12} = \nu_{13}$$

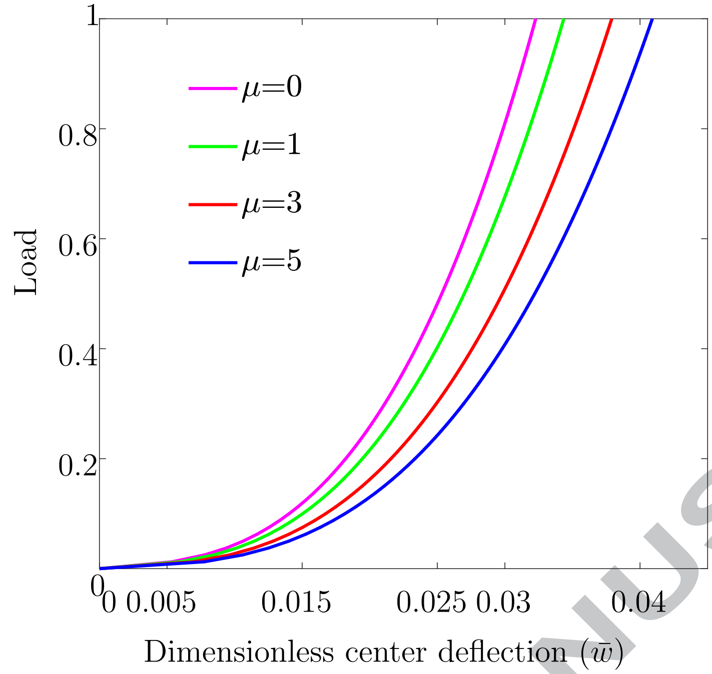

Figure 11: Load versus dimensionless center deflection plot of (-45°/45 °) plate with the SS-3 boundary conditions.

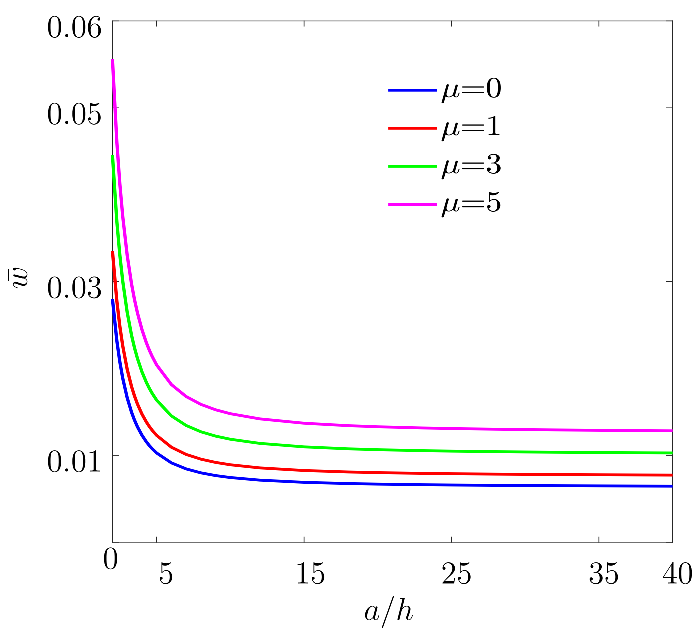

Figure 12: The dimensionless center deflection versus a/h with the SS-2 boundary conditions.

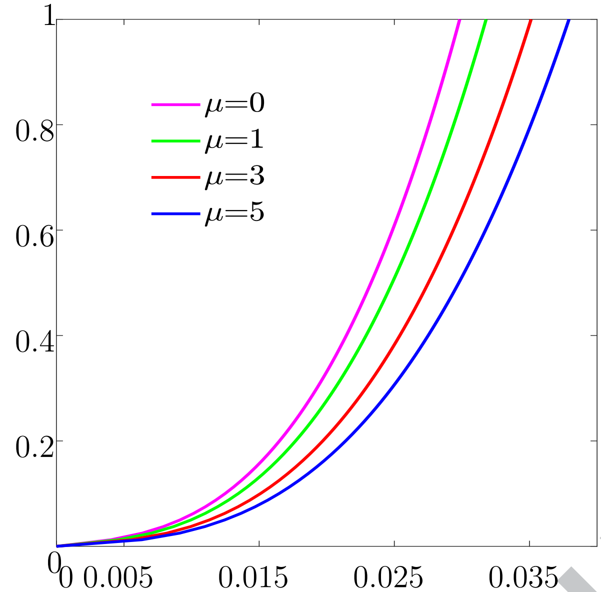

Figure 13: Load versus dimensionless center deflection plots of (-45°/45 °/-45°/45 °) plate with the SS-2 boundary conditions.

The load versus dimensionless center deflection ($\bar{w}$) plots of (-45°/45 °) plate with the SS-2 and SS-3 boundary conditions are presented in Figures 10 and 11, respectively. Figure 12 shows dimensionless center deflection versus a/h plot. The effect of the nonlocal parameter follows the same trends as discussed for the cross-ply lamination scheme.

Example 4

A four-layer square angle-ply (-45°/45/°45°/-45°) laminated plate subjected to transverse sinusoidal load of unit intensity is considered for the analysis. SS-2 and SS-3 type boundary conditions are considered. a/h ratio is taken as 10. The thickness of the each layer is considered equal. The following material properties are used:

$$E_1/E_2 = 25,\ G_{12}/E_2 = 0.6,\ G_{23}/E_2 = 0.5,\ G_{12} = G_{13},\ \nu_{12} = 0.25,\ \nu_{12} = \nu_{13}$$

Load versus dimensionless center deflection ($\bar{w}$) for (-45°/45 °/-45°/45 °) plates with the SS-2 and SS-3 boundary conditions are shown in Figures 13 and 14, respectively. It is clear that the SS-3 boundary condition make the plate stiffer compared to the SS-2 boundary conditions. It can also be observed that in the case of the four-layer (-45°/45 °/-45°/45 °) laminated plates the difference in the deflections predicted by the SS-2 and SS-3 boundary conditions is less compared to that of two-layer (-45°/45 °) laminated plates.

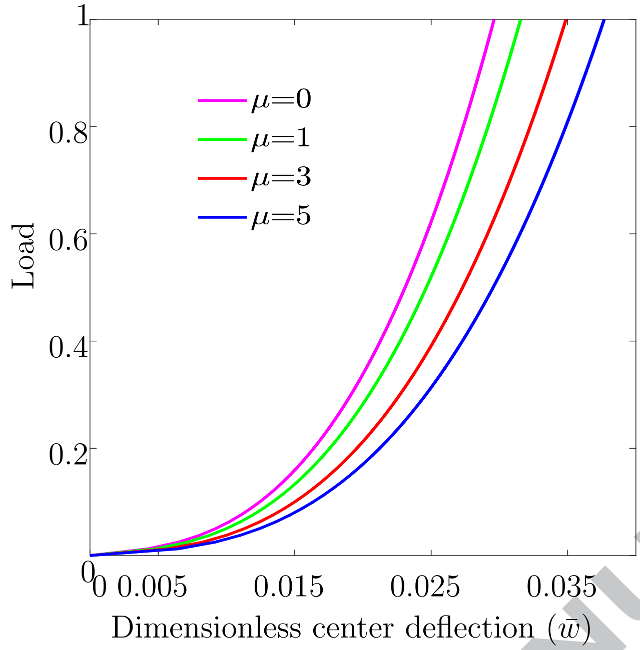

Figure 14: Load versus dimensionless center deflection plot of (-45°/45 °/-45°/45 °) plate with the SS-3 boundary conditions.

Conclusions

Equations of equilibrium of nonlocal Reddy's third-order shear deformation theory for the analysis of laminated composite plates are derived using Eringen's nonlocal differential constitutive equations and the von Kármán nonlinear strains. The weak forms are derived and the finite element model is developed. Examples with different stacking sequences and different boundary conditions are considered to illustrate the effect of the von Kármán nonlinearity, anisotropy, and nonlocality on the bending behaviour of laminated composite plates. It has been observed that Eringen's nonlocal model results in reduced structural stiffness of laminated composite plates and, therefore, the deflection increases with an increase in the nonlocal parameter.

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