Fractional cross intersecting families

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Abstract

Let $\mathcal{A} = \{A_1, ..., A_p\}$ and $\mathcal{B} = \{B_1, ..., B_q\}$ be two families of subsets of $[n]$ such that for every $i \in [p]$ and $j \in [q]$, $|A_i \cap B_j| = \frac{c}{d}|B_j|$, where $\frac{c}{d} \in [0, 1]$ is an irreducible fraction. We call such families $\frac{c}{d}$-*cross intersecting families.* In this paper, we find a tight upper bound for the product $|\mathcal{A}||\mathcal{B}|$ and characterize the cases when this bound is achieved for $\frac{c}{d} = \frac{1}{2}$. Also, we find a tight upper bound on $|\mathcal{A}||\mathcal{B}|$ when $\mathcal{B}$ is k-uniform and characterize, for all $\frac{c}{d}$, the cases when this bound is achieved.

1 Introduction

Let $[n]$ denote $\{1, ..., n\}$ and let $2^{[n]}$ denote the power set of $[n]$.We shall use $\binom{[n]}{k}$ to denote the set of all k-sized subsets of $[n]$. Let $\mathcal{F} \subseteq 2^{[n]}$. The family $\mathcal{F}$ is an *intersecting family* if every two sets in $\mathcal{F}$ intersect with each other. The famous Erdős-Ko-Rado Theorem [1] states that $|\mathcal{F}| \leq \binom{n-1}{k-1}$ if $\mathcal{F}$ is a k-uniform intersecting family, where $2k \leq n$. Several variants of the notion of intersecting families have been extensively studied in the literature. Given a set $L = \{l_1, \ldots, l_s\}$ of non-negative integers, a family $\mathcal{F} \subseteq 2^{[n]}$ is L-*intersecting* if for all $F_i, F_j \in \mathcal{F}, F_i \neq F_j, |F_i \cap F_j| \in L$. Ray-Chaudhuri and Wilson in [2] showed that if $\mathcal{F}$ is k-uniform and L-intersecting, then $|\mathcal{F}| \leq \binom{n}{s}$ and the bound is tight. Frankl and Wilson in [3] showed a tight upper bound of $\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$ if the restriction on the cardinalities of the sets of an L-intersecting family is relaxed. Further, if L is a singleton set, then Fisher inequality [4] gives an upper bound of $|\mathcal{F}| \leq n$ for the cardinality of an L-intersecting family $\mathcal{F}$. Recently, in [5], Balachandran et al. introduced a fractional variant of the classical L-intersecting families. For a survey on intersecting families, see [6].

Two families $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are *cross-intersecting* if $|A \cap B| > 0$, $\forall\ A \in \mathcal{A}, B \in \mathcal{B}$. Pyber in [7] showed that if $n \geq 2k$, and $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k}$ is a cross-intersecting pair of families, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}^2$. Frankl et al. in [8] showed that if $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ such that $|A \cap B| \geq t$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then for all $n \geq (t+1)(k-t+1)$, $|\mathcal{A}||\mathcal{B}| \leq \binom{n-t}{k-t}^2$, the cross-intersecting version of the Erdős-Ko-Rado Theorem. A cross-intersecting pair of families $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ is said to be l-cross-intersecting if $\forall A \in \mathcal{A},\ B \in \mathcal{B},\ |A \cap B| = l$, for some positive integer l. Ahlswede, Cai and Zhang showed in [9], for all $n \geq 2l$, a simple construction of an l-cross-intersecting pair $(\mathcal{A}, \mathcal{B})$ of families of subsets of $[n]$ with $|\mathcal{A}||\mathcal{B}| = \binom{2l}{l}2^{n-2l} = \Theta(\frac{2^n}{\sqrt{l}})$. Later Alon and Lubetzky in [10] showed that the $\Theta(\frac{2^n}{\sqrt{l}})$ bound is tight and characterized the cases when the bound is achieved.

In this paper, we introduce a fractional variant of the l-cross-intersecting families. Let $\mathcal{A} = \{A_1, ..., A_p\}$ and $\mathcal{B} = \{B_1, ..., B_q\}$ be two families of subsets of $[n]$ such that for every $i \in [p]$ and $j \in [q]$, $|A_i \cap B_j| = \frac{c}{d}|B_j|$, where $\frac{c}{d} \in [0, 1]$ is an irreducible fraction. We call such an $(\mathcal{A}, \mathcal{B})$ pair a $\frac{c}{d}$-cross-intersecting pair of families. Given c, d, and n, let $\mathcal{M}_{\frac{c}{d}}(n)$ denote the maximum value of $|\mathcal{A}||\mathcal{B}|$ where $(\mathcal{A}, \mathcal{B})$ is a $\frac{c}{d}$-cross intersecting pair of families of subsets of $[n]$. We have the following results:

Theorem 1.1. $\mathcal{M}_{\frac{c}{d}}(n) = 2^n$

When $\frac{c}{d} = 0$, $\mathcal{A} = 2^{[n]}$, $\mathcal{B} = \{\emptyset\}$ is a maximal pair. In fact, $\mathcal{A} = 2^{[k]}$, $\mathcal{B} = \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the power set of $S = \{k+1, \ldots, n\}$, are the only maximal pairs up to a relabelling of the elements, $0 \leq k \leq n$. When $\frac{c}{d} = 1$, $\mathcal{A} = \{[n]\}$ and $\mathcal{B} = 2^{[n]}$ is a maximal pair. In fact, $\mathcal{B} = 2^{[k]}$, $\mathcal{A} = \{A : A = [k] \cup T$, where $T \in \mathcal{P}(S)\}$, where $\mathcal{P}(S)$ is the power set of $S = \{k+1, \ldots, n\}$, are the only maximal pairs up to a relabelling of the elements, $0 \leq k \leq n$. In Theorem 1.2, we characterize all maximal pairs when $\frac{c}{d} = \frac{1}{2}$.

Theorem 1.2. *Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{1}{2}$-cross intersecting pair of families of subsets of $[n]$ with $|\mathcal{A}||\mathcal{B}| = 2^n$. Then $(\mathcal{A}, \mathcal{B})$ is one of the following $\lfloor \frac{n}{2} \rfloor + 1$ pairs of families $(\mathcal{A}_k, \mathcal{B}_k)$, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, up to isomorphism.*

$$\mathcal{A}_0 = 2^{[n]} \text{ and } \mathcal{B}_0 = \{\emptyset\}$$

$$\mathcal{A}_k = \{A \in 2^{[n]} : |A \cap \{2i-1, 2i\}| = 1 \quad \forall i, 1 \leq i \leq k\}$$

$$\mathcal{B}_k = \{B \in 2^{[n]} : |B \cap \{2i-1, 2i\}| \in \{0, 2\} \quad \forall i, 1 \leq i \leq k \text{ and } \forall j > 2k,\ j \notin B\},$$

where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

It would be interesting to show a characterization theorem for any $\frac{c}{d} \in [0,1]$. We do have such a general characterization theorem (along with a new tight upper bound) in Theorem 1.3 for the case when $\mathcal{B}$ is k-uniform. The proof is a direct application of Theorem 1.1 in [10].

Theorem 1.3. *Let $(\mathcal{A},\mathcal{B})$ be a $\frac{c}{d}$-cross intersecting pair of families of subsets of $[n]$. Let $\mathcal{B}$ be k-uniform. Then, there exists some $k_0 > 0$, such that for $k > k_0$ we have*

$$|\mathcal{A}||\mathcal{B}| \leq \binom{\frac{2ck}{d}}{\frac{ck}{d}} 2^{n-\frac{2ck}{d}}$$

and the bound is tight if and only if, either (a) or (b) hold:

(a) When $\frac{c}{d} = 1$, $\mathcal{A} = \{\{1,\ldots,\kappa\}\} \times 2^Y$, $\mathcal{B} = \binom{[\kappa]}{k}$ where $Y = \{\kappa+1,\ldots,n\}$ and $\kappa \in \{2k-1, 2k\}$ up to a relabelling of the elements of $[n]$.

(b) When $\frac{c}{d} \neq 1$:

(i) If k is even, $c = 1$, $d = 2$, $\frac{ck}{d} = \lceil \frac{k}{2} \rceil$,

(ii) If k is odd, $c = \frac{k+1}{2}$, $d = k$, $\frac{ck}{d} = \lceil \frac{k}{2} \rceil$,

and for both the cases((i) and (ii)), there exists some τ such that, $k + \tau \leq n$ and up to a relabelling of the elements of $[n]$,

$$\mathcal{A} = \{\cup_{T \in J} T : J \subset \{\{1,k+1\},\ldots,\{\tau,k+\tau\},\{\tau+1\},\ldots,\{k\}\}, |J| = \lceil \tfrac{k}{2} \rceil\} \times 2^X$$

where $X = \{k+\tau+1,\ldots,n\}$ and

$$\mathcal{B} = \{L \cup \{\tau+1,\ldots,k\} : L \subset \{1,\ldots,\tau,k+1,\ldots,k+\tau\}, |L \cap \{i,k+i\}| = 1 \text{ for all } i \in [\tau]\}.$$

2 Notations and definitions

Given any $S \subseteq [n]$, we shall use $\chi(S)$ to denote the *characteristic vector* of S which is a $0-1$ vector of size n having its i^{th} entry equal to 1 if and only if $i \in S$. The *weight* of a vector is the number of non-zero entries it has, and hence weight of $\chi(S)$ is the same as $|S|$.

For any family $\mathcal{A} \subseteq 2^{[n]}$, we shall (ab)use $\mathcal{A}$ to denote the collection of characteristic vectors of the members of $\mathcal{A}$ as well. The meaning will be clearly stated if not clear from the context.

Let V be a collection of vectors in $\mathbb{F}_2^n$. Then, we define the following:

1. $span(V)$: The collection of all the vectors that can be expressed as a linear combination in $\mathbb{F}_2$ of the vectors of V. We know that $span(V)$ is a vector space over $\mathbb{F}_2$.

2. $basis(V)$: We use $basis(V)$ to denote the basis of $span(V)$.

3. $dim(V)$: $dim(V) = |basis(V)|$

Definition 1. *$V \subseteq \mathbb{F}_2^n$ is a* linear code *if $V = span(V)$.*

Definition 2. *Given a linear code $C \subseteq \mathbb{F}_2^n$, the* dual code *$C^{\perp}$ is defined as,*

$$C^{\perp} = \{x \in \mathbb{F}_2^n | \langle x, c \rangle = 0, \forall c \in C\}$$

where $\langle x, y \rangle$ is the standard inner product over $\mathbb{F}_2$.

The following is a well-known fact that is easy to verify.

Lemma 2.1. *If $C \subseteq \mathbb{F}_2^n$ is a linear code, then $C^{\perp}$ is also a linear code.*

Definition 3. *Self orthogonal and self dual codes: A code C is self orthogonal if $C \subseteq C^{\perp}$ and it is self dual if $C = C^{\perp}$.*

3 Bounding $\mathcal{M}_{\frac{c}{d}}(n)$

Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{c}{d}$-cross-intersecting pair of families of subsets of $[n]$, where $\frac{c}{d} \in [0, 1]$ is an irreducible fraction. We shall (ab)use $\mathcal{A}, \mathcal{B}$ to denote the set of characteristic vectors of the sets in $\mathcal{A}, \mathcal{B}$ respectively. For any $a \in \mathcal{A}, b \in \mathcal{B}$, we observe that $\langle a, b \rangle \equiv |A \cap B| \pmod 2$, where $a = \chi(A)$, $b = \chi(B)$.

Partition the family $\mathcal{B}$ into two parts as,

$$\mathcal{B}_1 = \{B \in \mathcal{B} : |B| \equiv 0 \pmod{2d}\} \tag{1}$$
$$\mathcal{B}_2 = \{B \in \mathcal{B} : |B| \equiv d \pmod{2d}\} \tag{2}$$

As all the sets $B \in \mathcal{B}$ have their cardinality $|B|$ divisible by d, $\{\mathcal{B}_1, \mathcal{B}_2\}$ is a valid partition of $\mathcal{B}$. Therefore $\forall a \in \mathcal{A}$, $b \in \mathcal{B}$, using the $\frac{c}{d}$ intersection property, we have:

$$\langle a, b \rangle = \begin{cases} 1, if\ b \in \mathcal{B}_2\ and\ c\ is\ odd \\ 0,\ otherwise \end{cases}$$

Construction 1. *Construct a set $\mathcal{B}_1^{'}$, by appending a* 0 *to the left of every vector in $\mathcal{B}_1$, and a set $\mathcal{B}_2^{'}$ by appending a 1 to the left of every vector in $\mathcal{B}_2$. Let $\mathcal{B}^{'} = \mathcal{B}_1^{'} \cup \mathcal{B}_2^{'}$. Construct a set $\mathcal{A}^{'}$ by appending a* 1 *to the left of every vector in $\mathcal{A}$.*

We now have, the value of

$$\langle a, b\rangle = 0 \;\; \forall a \in \mathcal{A}^{'},\, b \in \mathcal{B}^{'}$$

So, $(span(\mathcal{A}^{'}), span(\mathcal{B}^{'}))$ is a pair of mutually orthogonal subspaces of $\mathbb{F}_2^{n+1}$ over $\mathbb{F}_2$. We thus have,

$$dim(span(\mathcal{A}^{'})) + dim(span(\mathcal{B}^{'})) \leq n+1$$

So, it follows that

$$\begin{aligned} |\text{span}(\mathcal{A}^{'})| \cdot |\text{span}(\mathcal{B}^{'})| &= 2^{\dim(span(\mathcal{A}^{'}))} \cdot 2^{\dim((span(\mathcal{B}^{'}))} \\ &= 2^{dim(span(\mathcal{A}^{'}))+dim(span(\mathcal{B}^{'}))} \\ &\leq 2^{n+1} \end{aligned} \tag{3}$$

Lemma 3.1. *If the elements of a linear code $C \subseteq \mathbb{F}_2^n$ are arranged as rows of a matrix M_C with n columns, then for each column, one of the following holds,*

(i) All the entries in that column are 0

(ii) Exactly half the entries in that column are 0*, and the rest are* 1*.*

Proof. As C is a linear code, if we pick any $a \in C$, and consider the set $S = \{a + x | x \in C\}$ where $a + x$ is the vector addition in $\mathbb{F}_2^n$, then by the definition of a linear code $S = C$. Let M_S be a matrix whose rows are the vectors of S, taken in any order. M_S and M_C have the same set of rows (only their order may differ).

Let $j \in [n]$. Column j in M_C and M_S have the same number of 1's(and 0's). Suppose (i) does not hold for column j in M_C. Then, some row, say a, in M_C has its j^{th} entry as 1. Let S, and thereby M_S, be defined according to this vector a. From the definition of S, it is clear that the number of 1's in the j^{th} column of M_S is equal to the number of 1's in the j^{th} column of M_C. Since adding a to any $\{0, 1\}$ vector flips the j^{th} coordinate of v, we conclude that (ii) holds for M_c. □

Corollary 3.2. $|span(\mathcal{A}^{'})| \geq 2|\mathcal{A}^{'}|$

Proof. The leftmost column of $\mathcal{M}_{\mathcal{A}^{'}}$ does not contain any 0. As $span(\mathcal{A}^{'})$ is a linear code and $\mathcal{A}^{'} \subseteq span(\mathcal{A}^{'})$, by condition (ii) of Lemma 3.1 above, $span(\mathcal{A}^{'})$ must have at least $|\mathcal{A}^{'}|$ more elements having their leftmost entry as 0. □

Now we prove the main result of this section which is Theorem 1.1.

Statement of Theorem 1.1**:** $\mathcal{M}_{\frac{c}{d}}(n) = 2^n$

Proof. $\mathcal{A} = 2^{[n]}$, $\mathcal{B} = \{\emptyset\}$ is a trivial example of a $\frac{c}{d}$ cross-intersecting pair of families having $|\mathcal{A}||\mathcal{B}| = 2^n$. Thus, $\mathcal{M}_{\frac{c}{d}}(n) \geq 2^n$. The proof of the upper bound for $\mathcal{M}_{\frac{c}{d}}(n)$ follows from Inequality (3) and Corollary 3.2. Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{c}{d}$ cross-intersecting pair of families of subsets of $[n]$. Let $\mathcal{A}'$, $\mathcal{B}'$ be constructed from $\mathcal{A}$, $\mathcal{B}$, respectively, as explained in the beginning of this section. Note that $|\mathcal{A}'| = |\mathcal{A}|$ and $|\mathcal{B}'| = |\mathcal{B}|$ by construction.

$$\begin{aligned}
2^{n+1} &\geq |\text{span}(\mathcal{A}')| \cdot |\text{span}(\mathcal{B}')| && \text{[from (3)]} \\
&\geq 2 \cdot |\mathcal{A}'| \cdot |\text{span}(\mathcal{B}')| && \text{[from Corollary 3.2]} \\
&\geq 2 \cdot |\mathcal{A}'| \cdot |\mathcal{B}'| \\
&= 2 \cdot |\mathcal{A}| \cdot |\mathcal{B}| && \text{[by construction]}
\end{aligned}$$

□

4 Characterization of maximal pairs when $\frac{c}{d} = \frac{1}{2}$

Definition 4. *Cross bisecting pair of families: A pair of families of subsets of $[n]$ is called a* cross-bisecting pair *if it is a $\frac{1}{2}$ cross-intersecting pair. $(\mathcal{A}, \mathcal{B})$ is called a* maximal *cross bisecting or simply a* maximal pair*, if it is a cross bisecting pair and $|\mathcal{A}||\mathcal{B}| = 2^n$.*

For example, $\mathcal{A} = 2^{[n]}$ and $\mathcal{B} = \{\emptyset\}$ is a trivial maximal pair. In this section, we characterize all maximal pairs. Let $(\mathcal{A}, \mathcal{B})$ be a cross bisecting pair and let $(\mathcal{A}', \mathcal{B}')$ be the associated pair constructed by appending bits as defined in the previous section.

Definition 5. *Let $f_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}'$ be a bijective mapping that maps every vector in $\mathcal{A}$ to its corresponding vector in $\mathcal{A}'$, and let $g_{\mathcal{A}} : \mathcal{A}' \to \mathcal{A}$ be its inverse. Likewise, define functions $f_{\mathcal{B}}$ and $g_{\mathcal{B}}$ between $\mathcal{B}$ and $\mathcal{B}'$. For any set $V \subseteq \mathcal{A}$, we shall use, $f_{\mathcal{A}}(V)$ to denote $\{f_{\mathcal{A}}(A)|\ A \in V\}$ and for any $V \subseteq \mathcal{A}'$, we use $g_{\mathcal{A}}(V)$ to denote $\{g_{\mathcal{A}}(A)|\ A \in V\}$. Similarly, for any $V \subseteq \mathcal{B}$, we use, $f_{\mathcal{B}}(V)$ to denote $\{f_{\mathcal{B}}(B)|\ B \in V\}$ and for any $V \subseteq \mathcal{B}'$, $g_{\mathcal{B}}(V)$ to denote $\{g_{\mathcal{B}}(B)|\ B \in V\}$*

Observation 1. *$f_{\mathcal{B}}(\mathcal{B}_1) = \mathcal{B}'_1$ and $f_{\mathcal{B}}(\mathcal{B}_2) = \mathcal{B}'_2$. Similarly, $g_{\mathcal{B}}(\mathcal{B}'_1) = \mathcal{B}_1$ and $g_{\mathcal{B}}(\mathcal{B}'_2) = \mathcal{B}_2$*

Suppose $(\mathcal{A},\mathcal{B})$ is a maximal pair. Then from the proof of Theorem 1.1, we must have :

$$|\text{span}(\mathcal{A}^{'})| = 2|\mathcal{A}^{'}| \quad (4)$$

$$|\text{span}(\mathcal{B}^{'})| = |\mathcal{B}^{'}| \quad (5)$$

$$\dim(span(\mathcal{A}^{'})) + \dim(span(\mathcal{B}^{'})) = n+1 \quad (6)$$

Proposition 4.1. *$\mathcal{B} = span(\mathcal{B})$. Further, $f_{\mathcal{B}}$ is a linear map.*

Proof. This follows from equation (5). Let $x_1, x_2 \in \mathcal{B}$. We show that $x_3 = x_1 + x_2 \in \mathcal{B}$. This would imply $\mathcal{B}$ is closed under addition in $\mathbb{F}_2^n$ over $\mathbb{F}_2$, and hence $\mathcal{B} = \text{span}(\mathcal{B})$.

Let $x_1^{'} = f_{\mathcal{B}}(x_1)$ and $x_2^{'} = f_{\mathcal{B}}(x_2)$. From Equation (5), we have, $w = x_1^{'} + x_2^{'} \in \mathcal{B}^{'}$. Since w and x_3 agree on each of the rightmost n bits of x_3, we have $g_{\mathcal{B}}(w) = x_3$. Since $w \in \mathcal{B}^{'}$, from the definition of the function $g_{\mathcal{B}}$ we have $x_3 = g_{\mathcal{B}}(w) \in \mathcal{B}$. Further, observe that $f_{\mathcal{B}}(x_1) + f_{\mathcal{B}}(x_2) = w = f_{\mathcal{B}}(x_3) = f_{\mathcal{B}}(x_1 + x_2)$ and hence $f_{\mathcal{B}}$ is a linear map. □

That $\mathcal{B}$ is a linear code from Proposition 4.1 implies closure of the family of subsets $\mathcal{B}$ under symmetric difference. In fact, we have the following stronger result.

Proposition 4.2. *Let vectors $b_1, b_2 \in \mathcal{B}$. Then, $b_1 + b_2 \in \mathcal{B}_1$ if and only if either b_1,$b_2 \in \mathcal{B}_1$, or b_1,$b_2 \in \mathcal{B}_2$. Otherwise, $b_1 + b_2 \in \mathcal{B}_2$.*

Proof. We prove the 2-way implication, and rest of the proposition follows from Proposition 4.1. Let $b_1^{'} = f_{\mathcal{B}}(b_1), b_2^{'} = f_{\mathcal{B}}(b_2)$.

- $b_1 + b_2 \in \mathcal{B}_1 \Rightarrow b_1$ and b_2 are both from $\mathcal{B}_1$, or both from $\mathcal{B}_2$
  Since $f_{\mathcal{B}}$ is a linear map, we have $(b_1 + b_2 \in \mathcal{B}_1) \Rightarrow (f_{\mathcal{B}}(b_1 + b_2) = f_{\mathcal{B}}(b_1) + f_{\mathcal{B}}(b_2) = b_1^{'} + b_2^{'} \in \mathcal{B}_1^{'})$. So, the leftmost bit of $b_1^{'} + b_2^{'}$ is 0. This means that the leftmost bit must be the same in $b_1^{'}$ and $b_2^{'}$, which directly implies that either $b_1^{'}$,$b_2^{'} \in \mathcal{B}_1^{'}$, or $b_1^{'}$,$b_2^{'} \in \mathcal{B}_2^{'}$.

- Either b_1,$b_2 \in \mathcal{B}_1$, or b_1,$b_2 \in \mathcal{B}_2 \Rightarrow b_1 + b_2 \in \mathcal{B}_1$
  Since $b_1^{'}$ and $b_2^{'}$ agree upon the leftmost bit, $b_1^{'} + b_2^{'}$ has a 0 in its leftmost bit. So, $b_1^{'} + b_2^{'} \in \mathcal{B}_1^{'}$. From the Observation 1 above, we have $b_1 + b_2 \in \mathcal{B}_1$.

□

Proposition 4.3. *$\mathcal{B}$ is a self-orthogonal code.*

Proof. We prove the proposition by showing that $\forall b_1, b_2 \in \mathcal{B}$, $\langle b_1, b_2 \rangle = 0$. Let B_1, B_2 be the sets corresponding to the vectors b_1, b_2, respectively. Since we are operating in the field $\mathbb{F}_2$, it is enough to show that $|B_1 \cap B_2|$ is even.

Let $b_3 = b_1 + b_2$. We observe that b_3 is the characteristic vector of $B_3 = B_1 \Delta B_2$, the symmetric difference of B_1 and B_2. We have,

$$|B_3| = |B_1 \Delta B_2| = |B_1| + |B_2| - 2|B_1 \cap B_2| \tag{7}$$

As $\frac{c}{d} = \frac{1}{2}$, $\forall B \in \mathcal{B}_1$, we have $|B| \equiv 0 \pmod 4$. By Proposition 4.1, $B_1 \Delta B_2 = B_3 \in \mathcal{B}$ as $\mathcal{B}$ is a linear code. Taking equation (7) modulo 4, if $B_3 \in \mathcal{B}_1$, then

$$|B_1| + |B_2| - 2|B_1 \cap B_2| \equiv 0 \pmod 4$$

By Proposition 4.2, both B_1 and B_2 are either from $\mathcal{B}_1$ or from $\mathcal{B}_2$. In both cases, $|B_1| + |B_2| \equiv 0 \pmod 4$ Therefore, $2|B_1 \cap B_2| \equiv 0 \pmod 4$ or $|B_1 \cap B_2| \equiv 0 \pmod 2$. If $B_3 \in \mathcal{B}_2$, then

$$|B_1| + |B_2| - 2|B_1 \cap B_2| \equiv |B_3| \equiv 2 \pmod 4$$

Again by Proposition 4.2, $|B_1| + |B_2| \equiv 2 \pmod 4$.
So, we have $2|B_1 \cap B_2| \equiv 0 \pmod 4$ or $|B_1 \cap B_2| \equiv 0 \pmod 2$. Thus in both cases, $|B_1 \cap B_2|$ is even, so $\mathcal{B}$ is a self-othogonal code. □

Lemma 4.4. *Let $(\mathcal{A}, \mathcal{B})$ be a maximal pair, then $|\mathcal{B}| \leq 2^{\lfloor \frac{n}{2} \rfloor}$*

Proof. It is a known result (see [11]) that for a linear code $C \subseteq \mathbb{F}_2^n$ and its dual code $C^\perp$,

$$\dim(C) + \dim(C^\perp) = n \tag{8}$$

For any self-orthogonal code C, $C \subseteq C^\perp$. So,

$$\dim(C) \leq \dim(C^\perp)$$

Applying equation (8) in this inequality, we get

$$n = \dim(C) + \dim(C^\perp) \geq 2\dim(C)$$
$$\text{Therefore, } \dim(C) \leq \frac{n}{2}$$

Since $\mathcal{B}$ is a self-orthogonal code (Proposition 4.3), we get $\dim(\mathcal{B}) \leq \frac{n}{2}$. Hence,

$$|\mathcal{B}| \leq 2^{\lfloor \frac{n}{2} \rfloor}$$

□

Proposition 4.5. *If a set A bisects B_1, B_2 and $B_1 \Delta B_2$, then A also bisects $B_1 \cap B_2$.*

Proof.

$$\begin{aligned}
|A \cap (B_1 \vartriangle B_2)| &= \frac{|B_1 \vartriangle B_2|}{2} \text{ [A bisects } B_1\Delta B_2] \\
\Rightarrow |A \cap ((B_1 \backslash B_2) \cup (B_2 \backslash B_1))| &= \frac{|B_1| + |B_2| - 2|B_1 \cap B_2|}{2} \\
\Rightarrow |A \cap (B_1 \backslash B_2)| + |A \cap (B_2 \backslash B_1)| &= \frac{|B_1|}{2} + \frac{|B_2|}{2} - |B_1 \cap B_2|
\end{aligned}$$

$$\Rightarrow |A \cap B_1| - |A \cap (B_1 \cap B_2)| + |A \cap (B_2)| - |A \cap (B_1 \cap B_2)| = \frac{|B_1|}{2} + \frac{|B_2|}{2} - |B_1 \cap B_2|$$

$$\begin{aligned}
\Rightarrow \frac{|B_1|}{2} + \frac{|B_2|}{2} - 2|A \cap (B_1 \cap B_2)| &= \frac{|B_1|}{2} + \frac{|B_2|}{2} - |B_1 \cap B_2| \\
&\quad \text{[since } A \text{ bisects both } B_1 \text{ and } B_2] \\
\Rightarrow 2|A \cap (B_1 \cap B_2)| &= |B_1 \cap B_2| \\
\Rightarrow |A \cap (B_1 \cap B_2)| &= \frac{|B_1 \cap B_2|}{2}
\end{aligned}$$

□

Proposition 4.6. *$\mathcal{B}$ is closed under intersection.*

Proof. Let B_1,$B_2 \in \mathcal{B}$. We show that $B_1 \cap B_2 \in \mathcal{B}$. By Proposition 4.1, $b_1 + b_2 \in \mathcal{B}$ i.e., $B_1 \Delta B_2 \in \mathcal{B}$. Let A be any arbitrary member of $\mathcal{A}$. Now, A bisects B_1, B_2 and $B_1 \Delta B_2$ as $(\mathcal{A}, \mathcal{B})$ is a cross bisecting pair. By Proposition 4.5, A bisects $B_1 \cap B_2$. Since $(\mathcal{A}, \mathcal{B})$ is a maximal pair, we conclude that $B_1 \cap B_2 \in \mathcal{B}$. □

Now, we prove the main result of this section,Theorem 1.2, the characterization of maximal pairs.

Statement of Theorem 1.2**:** *Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{1}{2}$-cross intersecting pair of families of subsets of $[n]$ with $|\mathcal{A}||\mathcal{B}| = 2^n$. Then $(\mathcal{A}, \mathcal{B})$ is one of the following $\lfloor \frac{n}{2} \rfloor + 1$ pairs of families $(\mathcal{A}_k, \mathcal{B}_k)$, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, up to isomorphism.*

$$\mathcal{A}_0 = 2^{[n]} \text{ and } \mathcal{B}_0 = \{\emptyset\}$$

$$\mathcal{A}_k = \{A \in 2^{[n]} : |A \cap \{2i-1, 2i\}| = 1 \quad \forall i, 1 \leq i \leq k\}$$

$$\mathcal{B}_k = \{B \in 2^{[n]} : |B \cap \{2i-1, 2i\}| \in \{0, 2\} \quad \forall i, 1 \leq i \leq k \text{ and } \forall j > 2k,\ j \notin B\},$$

where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

By isomorphism, it is meant that for any maximal pair $(\mathcal{A}, \mathcal{B})$, $\exists$ a bijective mapping $f : [n] \to [n]$ such that if every $A \in \mathcal{A}$ is replaced by $A_f = \{f(i) | i \in A\}$ and every $B \in \mathcal{B}$ is replaced by $B_f = \{f(i) | i \in B\}$ then the families $(\mathcal{A}_f, \mathcal{B}_f)$, where $\mathcal{A}_f = \{A_f | A \in \mathcal{A}\}$ and $\mathcal{B}_f = \{B_f | B \in \mathcal{B}\}$, is a maximal pair which is one of $(\mathcal{A}_k, \mathcal{B}_k)$, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Proof. Consider a maximal pair $(\mathcal{A}, \mathcal{B})$ where $\mathcal{B} \neq \{\emptyset\}$. We write the elements of $\mathcal{B}$ as rows of a $0-1$ matrix M_0. Suppose n_0 columns have only 0 entries in all the rows(n_0 may be 0). As the characterization is up to isomorphism, we may assume that these are the rightmost n_0 columns of the matrix M_0. In each of the remaining $n - n_0$ columns, from Lemma 3.1, there are exactly $\frac{|\mathcal{B}|}{2}$ 1's and $\frac{|\mathcal{B}|}{2}$ 0's as $\mathcal{B}$ is a linear code. (by Proposition 4.1)
Define

$$B_1 = \bigcap_{\substack{1 \in B, \\ B \in \mathcal{B}}} B$$

We write the $\frac{|\mathcal{B}|}{2}$ rows containing 1 in the leftmost column of M_0 as the top $\frac{|\mathcal{B}|}{2}$ rows to obtain a new matrix M_1 from M_0. And B_1 is one of these rows according to Proposition 4.6. Moreover, as all intersections are of even cardinality (Proposition 4.3), $|B_1|$ is even.
Let $|B_1| = 2i_1$, $i_1 \geq 1$. So, there are $2i_1 - 1$ elements in B_1 other than the element 1. Due to isomorphism, we may assume them to be $2, 3, \ldots, 2i_1$.
If $2i_1 + 1 \leq n - n_0$, then define the set B_2 as:

$$B_2 = \bigcap_{\substack{2i_1+1 \in B, \\ B \in \mathcal{B}}} B$$

Claim 4.7. $1 \notin B_2$

Proof. Assume for the sake of contradiction, $1 \in B_2$. This implies that for all the $\frac{|\mathcal{B}|}{2}$ sets which contain the element $2i_1 + 1$ also contain the element 1. From Lemma 3.1, (number of sets in $\mathcal{B}$ that contain the element 1) = (number of sets in $\mathcal{B}$ that contain the element $2i_1 + 1$) = $\frac{|\mathcal{B}|}{2}$. Hence, for any $B \in \mathcal{B}$, $1 \in B \iff 2i_1 + 1 \in B$. This implies that $2i_1 + 1 \in B_1$, which is a contradiction. Hence, $1 \notin B_2$ and therefore B_2 does not belong to the top $\frac{|\mathcal{B}|}{2}$ rows of M_1. □

Claim 4.8. $B_1 \cap B_2 = \emptyset$

Proof. Assume for the sake of contradiction, $x \in B_1 \cap B_2$. Then x is present in the $\frac{|\mathcal{B}|}{2}$ rows of the matrix M_1 whose intersection yields B_1. Since $x \in B_2$ and B_2 does not belong to these $\frac{|\mathcal{B}|}{2}$ rows of M_1 (by Claim 4.7). Thus, we have the element x present in at least $\frac{|\mathcal{B}|}{2} + 1$ rows of M_1, contradicting Lemma 3.1. □

We take the rows corresponding to the sets containing the $(2i_1+1)^{th}$ element that are not among the first $\frac{|\mathcal{B}|}{2}$ rows in M_1 and arrange them below the top $\frac{|\mathcal{B}|}{2}$ rows to create a matrix called M_2 from M_1. Again from Proposition 4.3, $|B_2|$ is even, say $2i_2$. Due to isomorphism and Claim 4.8, we may assume that $2i_1+1,\ldots,2i_1+2i_2$ are these $2i_2$ elements.

If $2i_1+2i_2+1 \le n-n_0$, then define,

$$B_3 = \bigcap_{\substack{2i_1+2i_2+1\in B,\\ B\in\mathcal{B}}} B$$

Claim 4.9. $1 \notin B_3$ *and* $2i_1+1 \notin B_3$.

The proof is similar to that of Claim 4.7

Claim 4.10. $B_1 \cap B_3 = \emptyset$ *and* $B_2 \cap B_3 = \emptyset$.

The proof is again similar to that of Claim 4.8.

We take the rows corresponding to the sets containing the $(2i_1+2i_2+1)^{th}$ element that are not among the first r rows ($r > \frac{|\mathcal{B}|}{2}$) in M_2 which contain the elements 1 or $2i_1+1$ and arrange them below the top r rows of M_2 to create a matrix called M_3 from M_2. From Proposition 4.3 and the definition of B_3, we have $|B_3| = 2i_3$, $i_3 \ge 1$. Due to isomorphism and Claim 4.10, we may assume that $2i_1+2i_2+1,\ldots,2i_1+2i_2+2i_3$ are these $2i_3$ elements.

We continue in this manner for k steps by constructing sets $B_1,\ldots,B_k$ and matrices $M_1,\ldots,M_k$, where $k \ge 1$, until we have $2i_1+\cdots+2i_k = n-n_0$. Observe that $B_1,\ldots,B_k$ and $P=\{n-n_0+1,\ldots,n\}$ is a partition of $[n]$.

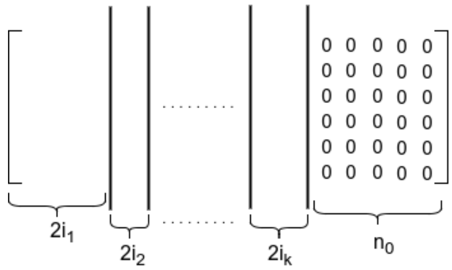

Figure 1: Partitioning the universe and thereby the columns of M_k

Claim 4.11. *For any set* $B \in \mathcal{B}$, $j \in [k]$, *we have* $B \cap B_j \in \{\emptyset, B_j\}$. *Further,* $B \cap P = \emptyset$.

Proof. From the definition of P, we have $B \cap P = \emptyset$. Let $j \in [k]$. Since B_j is equal to the intersection of some $\frac{|\mathcal{B}|}{2}$ sets in $\mathcal{B}$, we have B_j present as a subset of all these $\frac{|\mathcal{B}|}{2}$ sets. Applying Lemma 3.1, we can say that no element of B_j is present in any set in $\mathcal{B}$ other than these $\frac{|\mathcal{B}|}{2}$ sets. Hence, the claim. □

From Claim 4.11, observe that $S = \{B_1, \ldots, B_k\}$ forms a basis of the row space of the matrix M_k. The advantage of such a "disjoint basis" is that the bisection in one part is independent of another.

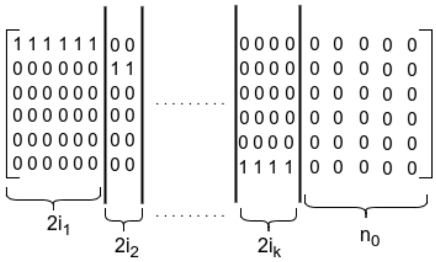

Figure 2: Basis for the code $\mathcal{B}$

Claim 4.12. *A set $A \in \mathcal{A}$ bisects every set in $\mathcal{B}$ if and only if it bisects every set in the basis S of $\mathcal{B}$.*

Proof. The forward direction is straightforward as $S \subseteq \mathcal{B}$. For the opposite direction, let $A \in \mathcal{A}$ be a set that bisects every member of S. Since the sets corresponding to the members in S are disjoint, any $B \in \mathcal{B}$ can be written as a union of some of these sets.
Let $B = B_1 \cup \cdots \cup B_l$, where $\{B_1, \ldots, B_l\} \subseteq S$. Then,

$$|A \cap B| = |A \cap (\bigcup_{j=1}^{l} B_j)| = \sum_{j=1}^{l} |A \cap B_j| = \sum_{j=1}^{l} \frac{|B_j|}{2} = \frac{|\bigcup_{j=1}^{l} B_j|}{2} = \frac{|B|}{2}$$

□

Since each set $A \in \mathcal{A}$ bisects the sets $B_1, \ldots, B_k$ and P, from Claim 4.12, the set A may contain any of the 2^{n_0} subsets of P, and $|A \cap B_1| = i_1, \ldots, |A \cap B_k| = i_k$. Since $\dim(\mathcal{B}) = k$, by Proposition 4.1, we have $|\mathcal{B}| = 2^k$.

$$|\mathcal{A}||\mathcal{B}| = \left(2^{n_0} \cdot \prod_{j=1}^{k} \binom{2i_j}{i_j}\right) \cdot 2^k \tag{9}$$

Recall that $\sum_{j=1}^{k} 2i_j = n - n_0$. Right hand side of Equation (9), is equal to 2^n if and only if $i_j = 1$, $\forall j \in [k]$.

Thus, if $\mathcal{B} \neq \{\emptyset\}$, then $(\mathcal{A}_k, \mathcal{B}_k)$, $k \geq 1$, defined in the statement of the theorem are the only maximal pairs. This completes the proof of Theorem 1.2.

□

5 Tight upper bound on $M_{\frac{c}{d}}(n)$ when $\mathcal{B}$ is k-uniform and characterization of the cases when the bound is achieved

Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{c}{d}$ cross-intersecting pair of families of subsets of $[n]$, where $\frac{c}{d} \in [0,1]$ is an irreducible fraction. In this section, we deal with the scenario when $\mathcal{B}$ is k-uniform, where $0 < k \leq n$. Since $\mathcal{B}$ is k-uniform, for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$, $|A \cap B| = \frac{ck}{d} = l$. Since c is relatively prime with d, and $|A \cap B|$ is an integer, we have k divisible by d. Therefore, we have a uniformly cross intersecting pair of families.

Alon and Lubetzky in [10] found a tight upper bound for the case of uniformly cross intersecting families and fully characterized the cases when the bound is achieved in the following theorem:

Theorem 5.1. *[Theorem 1.1 in [10]] There exists some $l_0 > 0$ such that, for all $l \geq l_0$, every l-cross intersecting pair $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ satisfies:*

$$|\mathcal{A}||\mathcal{B}| \leq \binom{2l}{l} 2^{n-2l}$$

Furthermore, if $|\mathcal{A}||\mathcal{B}| = \binom{2l}{l}2^{n-2l}$, then there exists some choice of parameters κ, τ, n':

$$\kappa \in \{2l-1, 2l\}, \tau \in \{0, \cdots, \kappa\}$$
$$\kappa + \tau \leq n' \leq n$$

such that upto a relabelling of the elements of $[n]$ and swapping $\mathcal{A}, \mathcal{B}$, the following holds:

$$\mathcal{A} = \{\bigcup_{T \in J} T : J \subset \{\{1, \kappa+1\}, \cdots, \{\tau, \kappa+\tau\}, \{\tau+1\}, \cdots, \{\kappa\}\}, |J| = l\} \times 2^X,$$
$$\mathcal{B} = \{L \cup \{\tau+1, \cdots, \kappa\} : L \subset \{1, \cdots, \tau, \kappa+1, \cdots, \kappa+\tau\}, |L \cap \{i, \kappa+i\}| = 1 \text{ for all } i \in [\tau]\} \times 2^Y$$

where $X = \{\kappa+\tau+1, \cdots, n'\}$ and $Y = \{n'+1, \cdots, n\}$.

Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{c}{d}$ cross-intersecting family where $\mathcal{B}$ is k-uniform. From Theorem 5.1, there exists a $k_0 > 0$ such that if $\frac{ck}{d} = l > k_0$, then $|\mathcal{A}||\mathcal{B}| \leq \binom{2l}{l}2^{n-2l}$. Consider the case when $\mathcal{B}$ corresponds to $\mathcal{B}$ of Theorem 5.1. If $|\mathcal{A}||\mathcal{B}| = \binom{2l}{l}2^{n-2l}$, then $n' = n$, $Y = \emptyset$, and $k = \kappa$ in the statement of Theorem 5.1. Since $l = \frac{ck}{d}$ and $k \in \{\frac{2ck}{d} - 1, \frac{2ck}{d}\}$, we have the following two cases:

Case 1: $k = \frac{2ck}{d} - 1$. Then, $(k+1)d = 2ck$. Since $gcd(c,d) = 1$ and $gcd(k, k+1) = 1$, we have $k|d|2k$. Thus, $d = k$ or $d = 2k$. We claim that $d = 2k$ is an invalid case.

This is because, when $d = 2k$, we have $c = k + 1$. Since $gcd(c, d) = 1$, k cannot be odd. And if k is even, then $l = \frac{ck}{d} = \frac{k+1}{2}$ is not an integer. So, the only valid case is $d = k$, $c = \frac{k+1}{2} = l$ and k is an odd integer.
Case 2: $k = \frac{2ck}{d}$. Then, $\frac{c}{d} = \frac{1}{2}$, that is $(\mathcal{A}, \mathcal{B})$ is a cross bisecting pair. Since $l = \frac{ck}{d} = \frac{k}{2}$ is an integer, k must be even in this case.
If $\mathcal{B}$ corresponds to $\mathcal{A}$ of Theorem 5.1, we have $X = \emptyset$, $\tau = 0$, $\mathcal{B}$ is $k(= l)$-uniform, $l = \frac{ck}{d}$. Thus, we have $\frac{c}{d} = 1$, $\mathcal{A} = \{\{1, \dots, \kappa\}\} \times 2^Y$ where $Y = \{\kappa + 1, \dots, n\}$ and $\mathcal{B} = \binom{[\kappa]}{k}$, $\kappa \in \{2k - 1, 2k\}$ up to a relabelling of the elements.

This leads us to the main result of this section.

Statement of Theorem 1.3**:** *Let* $(\mathcal{A},\mathcal{B})$ *be a* $\frac{c}{d}$*-cross intersecting pair of families of subsets of* $[n]$*. Let* $\mathcal{B}$ *be* k*-uniform. Then, there exists some* $k_0 > 0$*, such that for* $k > k_0$ *we have*

$$|\mathcal{A}||\mathcal{B}| \leq \binom{\frac{2ck}{d}}{\frac{ck}{d}} 2^{n - \frac{2ck}{d}}$$

and the bound is tight if and only if, either (a) *or* (b) *hold:*

(a) When $\frac{c}{d} = 1$*,* $\mathcal{A} = \{\{1, \dots, \kappa\}\} \times 2^Y$*,* $\mathcal{B} = \binom{[\kappa]}{k}$ *where* $Y = \{\kappa + 1, \dots, n\}$ *and* $\kappa \in \{2k - 1, 2k\}$ *up to a relabelling of the elements of* $[n]$*.*

(b) When $\frac{c}{d} \neq 1$*:*

(i) If k *is even,* $c = 1$*,* $d = 2$*,* $\frac{ck}{d} = \lceil \frac{k}{2} \rceil$*,*

(ii) If k *is odd,* $c = \frac{k+1}{2}$*,* $d = k$*,* $\frac{ck}{d} = \lceil \frac{k}{2} \rceil$*,*

and for both the cases((i) and (ii)), there exists some τ *such that,* $k + \tau \leq n$ *and up to a relabelling of the elements of* $[n]$*,*

$$\mathcal{A} = \{\cup_{T \in J} T : J \subset \{\{1, k+1\}, \dots, \{\tau, k+\tau\}, \{\tau+1\}, \dots, \{k\}\}, |J| = \lceil \tfrac{k}{2} \rceil\} \times 2^X$$

where $X = \{k + \tau + 1, \dots, n\}$ *and*

$$\mathcal{B} = \{L \cup \{\tau + 1, \dots, k\} : L \subset \{1, \dots, \tau, k+1, \dots, k+\tau\}, |L \cap \{i, k+i\}| = 1 \textit{ for all } i \in [\tau]\}.$$

6 Discussion

What are those pairs of $\frac{c}{d}$-cross intersecting families $(\mathcal{A}, \mathcal{B})$ which achieve $|\mathcal{A}||\mathcal{B}| = 2^n$ (equal to the upper bound for $\mathcal{M}_{\frac{c}{d}}(n)$ proved in Theorem 1.1)? In the introduction we characterize such families when $\frac{c}{d} = 0$ and $\frac{c}{d} = 1$. In Theorem 1.2, we characterize such families when $\frac{c}{d} = \frac{1}{2}$. From Theorem 1.3, we see that when $\mathcal{B}$ is k-uniform, $|\mathcal{A}||\mathcal{B}|$ is maximized when $\frac{c}{d}$ is 1 or nearly $\frac{1}{2}$($\frac{1}{2}$ or $\frac{1}{2} + \frac{1}{2k}$). For $\frac{c}{d} \in (0, 1)$, besides the case $\mathcal{A} = 2^{[n]}$, $\mathcal{B} = \{\emptyset\}$, is $|\mathcal{A}||\mathcal{B}| = 2^n$ achieved only when $\frac{c}{d}$ is close to $\frac{1}{2}$?

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