Degree conditions for forests in graphs

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Abstract

If H is any forest of order n with m edges, then any graph G of order $\geqslant n$ with $d(u)+d(v)\geqslant 2m-1$ for any two non-adjacent vertices u, v contains H.

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The Erdös–Sós conjecture that every graph with average degree greater than $(m-1)$ contains every tree with m edges, is one of the important problems in graph theory. In 1963, Erdös and Sós [5]. stated a conjecture on forests that any graph G of order n with

$$|E(G)| > \max\left\{\binom{2k-1}{2},\ \binom{k-1}{2} + (k-1)(n-k+1)\right\}$$

contains every forest with k edges and without isolated vertices as a subgraph. Brandt [4] proved this conjecture. He also proved that, if H is any forest of order n with m edges, then any graph of order $\geqslant n$ with minimum degree $\geqslant m$ contains H. These edge and minimum degree bounds are tight for matchings.

We prove that, if H is any forest of order n with m edges, then any graph of order $\geqslant n$ with $d(u)+d(v)\geqslant 2m-1$ for any two non-adjacent vertices u, v contains H. We show that, for some weaker degree conditions, graph G contains matching of size m but it does not contain all forests of size m. We prove that if H is any linear forest of order n with m edges, then any graph G of order $\geqslant n$ with at most i vertices of degree $\leqslant i$, for all $0 \leqslant i < m$, contains H.

Before proving the results let us fix the notations. All graphs considered are simple and finite. All terms that are not defined are standard and may be found in, for example, [3]. For a non-complete graph G, let $\sigma_2(G)$ be the minimum of $d(u) + d(v)$ over all pairs of non-adjacent vertices $u, v \in V(G)$, and $\sigma_2(G) = \infty$ when G is a complete graph. We say that a graph G contains a graph H if there is a subgraph of G isomorphic to H. If G contains H, and f is an isomorphism from H to a subgraph $f(H)$ of G, a vertex v of H is said to correspond to the vertex $f(v)$ of G. If H is any subgraph of G and v a vertex in G, then $d(v, H)$ is the number of vertices of H that are adjacent to v in G. If $v \notin V(H)$, $H + v$ is the subgraph of G obtained by adding to H the vertex v and all edges in G joining v to a vertex of H. If $V_s \subset V(G)$ then $G[V_s]$ denotes the subgraph induced by V_s. If G is any graph and S is either a vertex or edge in G, a subset of vertices or edges, or any subgraph of G, then $G - S$ is the subgraph of G obtained by deleting all vertices and edges in S.

Lemma 1. *Let T be any tree with $m \geqslant 1$ edges. Any graph G of order $\geqslant m + 1$ with $\sigma_2(G) \geqslant 2m - 1$ contains T.*

Proof. We prove it by induction on the number of edges m of T. If $m=1$ then G should contain at least one edge. Let xy be an edge of T such that y is a leaf. Let $T_1 = T - y$. Let w be a vertex of minimum degree in G. Let $G_1 = G - w$. Since removal of w from G can reduce the degree of any vertex by at most one, $\sigma_2(G_1) \geqslant 2(m-1) - 1$. By induction hypothesis, G_1 contains a subgraph T_1', which is isomorphic to T_1. Assume that $u \in V(T_1')$ corresponds to $x \in V(T_1)$. If u has any neighbour $v \in V(G) \setminus V(T_1')$, add vertex v and edge uv to T_1', to obtain a subgraph of G isomorphic to T. If u has no neighbour in $V(G) \setminus V(T_1')$ then $d(u, G) < m$, since $|T_1'| = m$. Since w is a minimum degree vertex in G, $d(w, G) \leqslant d(u, G) < m$. So $d(u, G) + d(w, G) < 2m - 1$, this contradicts with $\sigma_2(G) \geqslant 2m - 1$, since w is not adjacent to u. $\square$

Lemma 2. *Let T be any non-trivial subtree of a graph G with $|T| = t$. Let u, v be two vertices in $V(G) \setminus V(T)$ such that $d(u, T) + d(v, T) \geqslant 2t - 1$. Then there exists a neighbour w of v in T such that $T + u - w$ contains T.*

Proof. If $d(u, T) = t$, we can choose w to be any neighbour of v in T. Since u is adjacent to all the neighbours of w in T, we can replace w by u in T. If $d(u, T) = t - 1$, then $d(v, T) = t$ and we can choose w to be the vertex of T that is not adjacent to u. $\square$

Theorem 1. *Let F be any forest with m edges. Any graph G of order $\geqslant |F|$ with $\sigma_2(G) \geqslant 2m - 1$ contains F.*

Proof. Without loss of generality assume that every component of F is a non-trivial tree. We prove it by induction on the number of components of F. If F is a tree, it follows from Lemma 1. Let $T_1, T_2, \ldots, T_k$ be the components of F. Let T_1^s be a subgraph of G isomorphic to T_1 such that the number of edges in $G[V(T_1^s)]$ is maximum. Let $G[V(T_1^s)]$ be G_1 and $G_2 = G - G_1$. If $\sigma_2(G_2) \geqslant 2(m - |T_1| + 1) - 1$, G_2 contains $F - T_1$, by induction hypothesis, and hence G contains F. So assume $\sigma_2(G_2) < 2(m - |T_1| + 1) - 1$.

Suppose G_1 is not a complete graph. Let $v \in V(G_1)$ such that $d(v, G_1) < |T_1| - 1$. Let $u \in V(G_2)$ be a vertex that is adjacent to every vertex in G_1. Such a vertex should exist as $\sigma_2(G_2) < 2(m - |T_1| + 1) - 1$ but $\sigma_2(G) \geqslant 2m - 1$. The graph $G_1 - v + u$ contains T_1 as a spanning tree and has more edges than G_1, which is a contradiction. So, G_1 should be a complete graph.

Let z be any vertex in G_1. Let $G_1^s = G_1 - z$ and $G_2^s = G_2 + z$. Since $\sigma_2(G_2^s) \geqslant 2(m - |T_1| + 1) - 1$ and $|G_2^s| \geqslant |F| - |T_1| + 1$, by induction hypothesis G_2^s contains a subgraph isomorphic to $F - T_1$, let it be F^s. Let $G_3 = G_2^s - F^s$. If there is an edge between a vertex x of G_1^s and a vertex y of G_3, $G_1^s + y$ contains a subgraph isomorphic to T_1. F^s along with this subgraph gives the required subgraph isomorphic to F in G. If $\delta(G_3) \geqslant |T_1| - 1$, G_3 contains T_1, by Lemma 1.

Let $u \in V(G_3)$ such that $d(u, G_3)$ is minimum. Let v be any vertex in G_1^s. Let T_j^s be the component of F^s isomorphic to the component T_j of F, for all $2 \leqslant j \leqslant k$. There should be a component T_i^s of F^s, where $2 \leqslant i \leqslant k$, such that $d(u, T_i^s) + d(v, T_i^s) \geqslant 2(|T_i| - 1) + 1$, since $d(u, G) + d(v, G) \geqslant 2m - 1$ but $d(v, G_1^s) + d(u, G_3) \leqslant 2(|T_1| - 2)$. By Lemma 2, there exists a neighbour w of v in T_i^s such that $T_i^s - w + u$ contains T_i. Since $G_1^s + w$ contains T_1, we obtain a subgraph isomorphic to F in G. □

Now we will look at some other weaker degree conditions. These are motivated by the corresponding results for Hamiltonian cycles [2].

The Komlós–Sós [1] conjecture states that any graph G of order n with at least half of its vertices of degree at least k contains all trees of size k. This conjecture cannot be generalized to forests since $K_{2m-1} \cup K_1$, where $m \geqslant 2$, does not contain a matching of size m.

Let us look at a Pósá-type degree condition. Suppose the graph contains at most i vertices of degree $\leqslant i$, for $0 \leqslant i < m$. The graph G obtained by adding an edge between disjoint copies of K_{3m+1} and K_2 does not contain three disjoint stars, each of order $m + 1$, even though G contains at most i vertices of degree $\leqslant i$, for $1 \leqslant i < 3m$. So we will look at linear forests, i.e. forests in which every component is a path.

Theorem 2. *Let F be any linear forest with m edges and k components. Let G be any graph with $\geqslant |F|$ vertices. If there are at most i vertices in G with degree $\leqslant i$, for $0 \leqslant i < m$, then G contains F.*

Proof. Without loss of generality assume that every component of F is a non-trivial path. Since $|G| \geqslant |F|$, if G contains a Hamiltonian path, G contains F. Take a new vertex v and add edges between v and every vertex in G; let this new graph be G^s. In G^s the number of vertices of degree $\leqslant k$ is $< k$, for $1 \leqslant k \leqslant m$, and G^s is a connected graph.

Consider a longest path P in G^s such that the sum of the degrees of the end points is maximum. Suppose that the degree of one end point u is $d \leqslant m$. Then for every neighbour of u in P, the vertex preceding it in P must have degree $\leqslant d$; otherwise we obtain another longest path with that vertex as the end point. Thus we obtain $\geqslant d$ vertices of degree $\leqslant d$, a contradiction. So the degrees of both the end points are $\geqslant m + 1$.

Suppose $|P| \leqslant 2m+2$. Let $P = v_1, v_2, \dots, v_k$. Then we can find $v_i, v_{i+1} \in V(P)$ such that $v_1 v_{i+1}, v_k v_i \in E(G^s)$. So the subpaths $v_1, v_2, \dots, v_i$ and $v_{i+1}, v_{i+2}, \dots, v_k$ along with the edges $v_1 v_{i+1}, v_i v_k$ form a cycle of order $|P|$. Since G^s is connected, this contradicts the assumption that G does not contain a Hamiltonian path and P is a longest path in G^s. So $|P| \geqslant 2m+3$.

Since F contains m edges and k components, the order of F is $m+k$. So the order of a longest path in F is $\leqslant m-k+2$. Let $P_1, P_2, \dots, P_k$ be the components of F. Let $F_1 = F \cup P_{k+1}$, where P_{k+1} is a path of order $m-k+2$ and disjoint from F. So $|F_1| \leqslant |P|$, and thus P contains a subgraph F_1' isomorphic to F_1. Let the component P_i' of F_1' correspond to the component P_i of F_1, for $1 \leqslant i \leqslant k+1$.

If the newly added vertex $v \in V(P_{k+1}')$, then G contains F, since $F_1 = F \cup P_{k+1}$. If $v \in V(P_i')$, where $1 \leqslant i \leqslant k$, then G contains $F_1 - P_i$. Since the order of P_{k+1} is greater than or equal to the order of a largest component in F, G contains F. $\square$

Another possible generalization of Theorem 1 is to consider the closure of a graph, as defined by Bondy and Chvátal. This generalization cannot be applied to forests in general since $2K_{1,3}$ does not contain a path of length 3, but adding an edge between the centers of the stars gives a path of length 3.

Theorem 3. *Let F be any forest with m edges such that each component is a star. Let G be any graph and u, v be two non-adjacent vertices in it such that $d(u,G)+d(v,G) \geqslant 2m-1$. G contains F iff $G+uv$ contains F.*

Proof. Suppose every subgraph of $G+uv$ isomorphic to F includes the edge uv. Let $S_1, S_2, \dots, S_k$ be the components of F. Let F' be a subgraph of $G+uv$ isomorphic to F and let S_i' be the component of F' isomorphic to the component S_i of F, for $1 \leqslant i \leqslant k$.

Assume that $uv \in E(S_k')$ and u is the centre of S_k'. Let $F_1 = G - (F' - S_k')$. If $\Delta(F_1) \geqslant |S_k| - 1$, then F_1 contains a subgraph isomorphic to S_k, which along with $F' - S_k'$ gives a subgraph of G isomorphic to F. So $d(u, F_1) = |S_k| - 2$ and $d(v, F_1) \leqslant |S_k| - 2$. Then we can find a component S_i' of F', where $1 \leqslant i < k$, such that $d(u, S_i') + d(v, S_i') \geqslant 2(|S_i| - 1) + 1$, since $d(u,G) + d(v,G) \geqslant 2m-1$. By Lemma 2, we can find a vertex w in S_i' such that w is adjacent to u and $S_i' - w + v$ contains S_i'. $\square$

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