A NOTE ON FOURIER-JACOBI COEFFICIENTS OF SIEGEL MODULAR FORMS

SANOLI GUN AND NARASIMHA KUMAR

Abstract. Let F be a Siegel cusp form of weight k and genus $n > 1$ with Fourier-Jacobi coefficients f_m. In this article, we estimate the growth of the Petersson norms of f_m, where m runs over an arithmetic progression. This result sharpens a recent result of Kohnen in [5].

1. Introduction

Let $\mathcal{H}_n$ be the Siegel upper half-plane of genus $n \geq 1$ and $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$ be the full Siegel modular group. Also let $S_k(\Gamma_n)$ be the space of Siegel cusp forms of weight k on Γ_n.

For $Z \in \mathcal{H}_n$, write $Z = \left(\begin{smallmatrix} \tau & z^t \\ z & \tau' \end{smallmatrix}\right)$, where $\tau \in \mathcal{H}_{n-1}$, $z \in \mathbb{C}^{n-1}$ and $\tau' \in \mathcal{H}_1$. If $F \in S_k(\Gamma_n)$ with $n > 1$, the Fourier-Jacobi expansion of F relative to the maximal parabolic group of type $(n-1, 1)$ is of the form

$$F(Z) = \sum_{m \geq 1} f_m(\tau, z) e^{2\pi i m \tau'}.$$

The functions f_m belong to the space $J_{k,m}^{\mathrm{cusp}}$ of Jacobi cusp forms of weight k, index m and of genus $n-1$, i.e., invariant under the Jacobi group $\Gamma_{n-1}^J := \Gamma_{n-1} \ltimes \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$. For $f, g \in J_{k,m}^{\mathrm{cusp}}$, the inner product of f and g is defined by

$$\langle f, g \rangle = \int_{\Gamma_{n-1}^J \backslash \mathcal{H}^{n-1} \times \mathbb{C}^{n-1}} f(\tau, z)\overline{g(\tau, z)} (\det v)^{k-n-1} e^{-4\pi m v^{-1}[y^t]} du dv dx dy,$$

where $\tau = u + iv, z = x + iy$.

Let $a, q \geq 2$ be natural numbers with $(a, q) = 1$. In [1, Thm. 1], Böcherer, Bruinier and Kohnen showed that for any non-zero function F in $S_k(\Gamma_n)(n > 1)$ with Fourier-Jacobi coefficients f_m, there exist infinitely many $m \in \mathbb{N}$ with $m \equiv a \pmod{q}$ such that $\langle f_m, f_m \rangle \neq 0$. In this article, we prove the existence of infinitely many $m \in \mathbb{N}$ with $m \equiv a \pmod{q}$ such that $\langle f_m, f_m \rangle > c_{F,q} m^{k-1}$ (see Theorem 3.1). This also improves a recent result of Kohnen [5] about existence of infinitely many $m \geq 1$ such that $\langle f_m, f_m \rangle > c_F m^{k-1}$. In order to prove our result, we combine the techniques of [1] and [5].

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2. Preliminaries

Let F be a non-zero cusp form in $S_k(\Gamma_n)(n > 1)$ with Fourier-Jacobi coefficients $\{f_m\}_{m\in\mathbb{N}}$. By the works of Kohnen and Skoruppa [2] and of Krieg [6], we know that $\langle f_m, f_m\rangle \ll_F m^k$ (the constant in $\ll$ depends only on F). Hence for natural numbers a, q with $(a, q) = 1$, the Dirichlet series

$$D(s; a, q, F) := \sum_{\substack{m\geq 1\\ m\equiv a \bmod q}} \frac{\langle f_m, f_m\rangle}{m^s}$$

converges for $s \in \mathbb{C}$ with $\Re(s) > k + 1$.

Proposition 2.1. *Let $a, q > 1$ be natural numbers with $(a, q) = 1$ and $F \in S_k(\Gamma_n)$, where $n > 1$. Then the Dirichlet series $D(s; a, q, F)$ converges for $\Re(s) > k$ and has a simple pole at $s = k$. Moreover, it vanishes at $s = 0, -1, -2, \cdots$.*

Proof. Let χ be a Dirichlet character modulo q. Then the Dirichlet series

$$D(s, \chi, F) := \sum_{m\geq 1} \frac{\chi(m)\langle f_m, f_m\rangle}{m^s}$$

converges for $s \in \mathbb{C}$ with $\Re(s) \gg 0$. Let χ_0 be the principal Dirichlet character modulo q. For $\chi \neq \chi_0$, we know that the completed Dirichlet series

$$D^*(s, \chi, F) := \left(\frac{2\pi}{q}\right)^{-2s} \Gamma(s)\Gamma(s-k+n)L(2s-2k+2n, \chi^2)\ D(s, \chi, F)$$

extends to a holomorphic function on $\mathbb{C}$ (see [4] and [5] for details). But when $\chi = \chi_0$, the completed Dirichlet series $D^*(s, \chi_0, F)$ has a meromorphic continuation to $\mathbb{C}$ with a simple real pole at $s = k$ (see [4, page 495] and the remark in page 7 of [1]).

We know if $\chi^2 \neq \chi_0$, then the real zeros of $L(s, \chi^2)$ are at $s = 0, -2, -4, \cdots$ since χ^2 is an even character. Also the poles of $\Gamma(s)$ are at $s = 0, -1, -2, \cdots$. Further, all these zeros and poles are simple. Hence $D(s, \chi, F)$ for $\chi \neq \chi_0$ extends to a holomorphic function on $\mathbb{C}$ and vanishes at $s = 0, -1, -2, \cdots$.

If $\chi = \chi_0$, then $D(s, \chi_0, F)$ has a meromorphic continuation to $\mathbb{C}$ possibly with a simple pole at $s = k$. Indeed, the function $D(s, \chi_0, F)$ has a simple real pole at $s = k$, since $D^*(s, \chi_0, F)$ has a simple real pole at $s = k$ and none of the functions $L(2s - 2k + 2n, \chi^2)$, $\Gamma(s - k + n)$ and $\Gamma(s)$ have a zero or a pole at $s = k$ and they are holomorphic there. Furthermore, the series $D(s, \chi_0, F)$ vanishes at $s = 0, -1, -2, \cdots$.

Hence using orthogonality of characters, we get

$$\begin{aligned} D(s; a, q, F) &= \frac{1}{\varphi(q)} \sum_{m\geq 1} \sum_{\chi \bmod q} \chi(a^{-1}m)\langle f_m, f_m\rangle m^{-s} \\ &= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \chi(a^{-1})\ D(s, \chi, F) \quad \text{for } \Re(s) > k. \qquad (1) \end{aligned}$$

This implies that the Dirichlet series $D(s; a, q, F)$ has a meromorphic continuation to $\mathbb{C}$ with a simple real pole at $s = k$ and vanishes at $s = 0, -1, -2, \cdots$. □

Remark 2.2. It is clear from equation (1) that the residue of $D(s; a, q, F)$ at $s = k$ depends only on q and the residue of $D(s, \chi_0, F)$ at $s = k$, but not on a. In fact, the residue of $D(s, \chi_0, F)$ can be expressed in terms of the Petersson scalar product of F with the Trace of "χ_0-twist of F" (see [4, Thm. 1] for further details).

Remark 2.3. The residue of $D(s; a, q, F)$ at $s = k$ is real and positive. This is true as $D(s; a, q, F)$ is holomorphic on $\Re(s) > k$ with a simple pole at $s = k$ and is positive on the real half axis $\Re(s) > k$.

We end this section by recalling a recent result of Pribitkin on Dirichlet series with oscillating coefficients.

Definition. We call a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \in \mathbb{R}$ oscillatory if there exist infinitely many n such that $a_n > 0$ and infinitely many n such that $a_n < 0$.

Theorem 2.4 (Pribitkin [7],[8]). *Let a_n be a sequence of real numbers such that the associated Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be non-trivial and it converges on some half-plane. If $F(s)$ is holomorphic on the whole real line and has infinitely many real zeros, then the sequence $\{a_n\}_{n=1}^{\infty}$ is oscillatory.

3. Statement and proof of the Main Result

Let $a, q > 1$ be natural numbers with $(a, q) = 1$. Also let $c_{F,q}$ be the residue of the function $D(s; a, q, F)$ at $s = k$. Recall that $c_{F,q}$ is independent of a by Remark 2.2. Moreover, $c_{F,q}$ is real and positive by Remark 2.3.

Theorem 3.1. *Let $a, q > 1$ be natural numbers with $(a, q) = 1$ and F be a non-zero Siegel cusp form in $S_k(\Gamma_n)$, $n > 1$ with Fourier-Jacobi coefficients $\{f_m\}_{m\in\mathbb{N}}$. Then there exist infinitely many m with $m \equiv a \bmod q$ such that $\langle f_m, f_m\rangle > c_{F,q} m^{k-1}$.*

Proof. Consider the Dirichlet series

$$\overline{D}(s; a, q, F) = D(s; a, q, F) - c_{F,q}\zeta(s - k + 1) \quad \text{for } \Re(s) > k. \tag{2}$$

By Proposition 2.1, the series $\overline{D}(s; a, q, F)$ has a meromorphic continuation to $\mathbb{C}$ with no poles on the real line and vanishes at $s = k - 1 - 2t$, where $t \in \mathbb{N},\ t > (k - 1)/2$.

For $m \geq 1$, let

$$\beta(m) \ := \begin{cases} \langle f_m, f_m\rangle - c_{F,q} m^{k-1} & \text{if } m \equiv a \pmod{q} \\ -c_{F,q} m^{k-1} & \text{otherwise} \end{cases} \tag{3}$$

be the general coefficient of $\overline{D}(s;a,q,F)$. We know that $\overline{D}(s;a,q,F)$ cannot be identically zero as $c_{F,q} > 0$ by Remark 2.3. Then by using Theorem 2.4, there exist infinitely many m with $m \equiv a \pmod q$ such that $\langle f_m, f_m\rangle > c_{F,q} m^{k-1}$. □

Using the above method, we are unable to prove that there exist infinitely many m with $m \equiv a \pmod q$ such that $\langle f_m, f_m\rangle < c_{F,q} m^{k-1}$. But we can prove the following weaker theorem.

Theorem 3.2. *Let F be a non-zero cusp form in $S_k(\Gamma_n), n > 1$ with Fourier-Jacobi coefficients $\{f_m\}_{m\in\mathbb{N}}$. Let q be a natural number and also let $c_{F,q}$ be the residue of $D(s;a,q,F)$ for some $a \in \mathbb{N}$ with $(a,q) = 1$ (hence for all a). Then there exist natural numbers b, c with $(bc, q) = 1$ such that the following hold:*

- *there exist infinitely many $m \in \mathbb{N}$ with $m \equiv b \pmod q$ such that $\langle f_m, f_m\rangle > q c_{F,q} m^{k-1}$ and*
- *there exist infinitely many $m \in \mathbb{N}$ with $m \equiv c \pmod q$ such that $\langle f_m, f_m\rangle < q c_{F,q} m^{k-1}$.*

Proof. Consider the Dirichlet series

$$\overline{D}(s;a,q,F) := \sum_{\substack{a=1\\(a,q)=1}}^{q-1} D(s;a,q,F) \; - \; \alpha_{F,q} M \zeta(s-k+1),$$

where

$$M := \prod_{\substack{p\mid q\\ p \text{ prime}}} (1 - p^{-(s-k+1)}) \quad \text{and} \quad \alpha_{F,q} := q c_{F,q}.$$

Since

$$M\zeta(s-k+1) = q^{-(s-k+1)} \sum_{\substack{a=1\\(a,q)=1}}^{q-1} \zeta(s-k+1, a/q),$$

we have

$$\begin{aligned}\overline{D}(s;a,q,F) &= \sum_{\substack{a=1\\(a,q)=1}}^{q-1} \left[D(s;a,q,F) \; - \; \alpha_{F,q} q^{-(s-k+1)} \zeta(s-k+1, a/q)\right] \\ &= \sum_{\substack{a=1\\(a,q)=1}}^{q-1} \sum_{\substack{m\geq 1\\ m\equiv a \bmod q}} \frac{\langle f_m, f_m\rangle - \alpha_{F,q} m^{k-1}}{m^s}.\end{aligned}$$

By Proposition 2.1, the series $\overline{D}(s;a,q,F)$ has a meromorphic continuation to $\mathbb{C}$ with no poles on the real line and vanishes at $s = k-1-2t$, where $t \in \mathbb{N},\ t > (k-1)/2$.

Note that the function $\overline{D}(s;a,q,F)$ can not be identically zero. If otherwise,

$$\sum_{\substack{a=1\\(a,q)=1}}^{q-1} D(s;a,q,F) = \alpha_{F,q} M \zeta(s-k+1).$$

This is a contradiction as zeros of the Riemann zeta function on the negative real axis are at negative even integers whereas each $D(s; a, q, F)$ has zeros at all negative integers. Now using Theorem 2.4, we get the desired result. □

As an immediate corollary, we get

Corollary 3.3. *For $n > 1$, let F be a non-zero cusp form in $S_k(\Gamma_n)$ with Fourier-Jacobi coefficients $\{f_m\}_{m\in\mathbb{N}}$. Let $c_{F,2}$ be the residue of the series $D(s; 1, 2, F)$ at $s = k$. Then the following hold:*

- *there exist infinitely many odd $m \in \mathbb{N}$ such that $\langle f_m, f_m\rangle > 2c_{F,2}m^{k-1}$ and*
- *there exist infinitely many odd $m \in \mathbb{N}$ such that $\langle f_m, f_m\rangle < 2c_{F,2}m^{k-1}$.*

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Sanoli Gun ,
Institute of Mathematical Sciences, C.I.T Campus, Taramani, Chennai 600113, India.
E-mail address: sanoli@imsc.res.in

Narasimha Kumar
Indian Institute of Technology Hyderabad, Ordnance Factory Estate, Yeddumailaram 502205, India.
E-mail address: narasimha.kumar@iith.ac.in